Lecture 24 – October 22, 2018

Announcements

- Midterm 2 average 92.50/122 (75.82%)

Today

- Definition of the definite integral
- Computing definite integrals as limits of Riemann Sums
- Properties of the definite integral
- Definite integrals from area formulas or symmetry

Definite Integral

1. Our definition of Riemann Sum is makes sense even if $f(x) < 0$ on the interval $[a, b]$.
2. What do we get?
3. An approximation of **net area**. That is area above x-axis minus area below x-axis
4. **Net area** is often more useful (and a more fundamental notion) than area

Definite Integral

Definition

The **mesh** of a partition $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ of the interval $[a, b]$ is

$$\begin{aligned} \Delta &= \max_{1 \leq k \leq n} \{\Delta x_k\} \\ &= \max_{1 \leq k \leq n} \{x_k - x_{k-1}\} \end{aligned}$$

In other words, the mesh of a partition is the width of the widest subinterval of the partition.

Definition (Definite Integral)

The function f is **integrable** on the interval $[a, b]$ if

$$\lim_{\Delta \to 0} \sum_{i=1}^{n} f(x_k^*) \Delta x_k$$

exists. If f is integrable on $[a, b]$ then the **definite integral** is

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

Theorem (Definite integral from regular Riemann Sums)

If f is integrable on $[a, b]$ then the **definite integral** *satisfies*

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n \to \infty} L_n = \lim_{n \to \infty} R_n = \lim_{n \to \infty} M_n$$

where L_n is the nth regular left Riemann Sum, R_n is the nth regular right Riemann Sum and M_n is the nth regular midpoint Riemann Sum.

Definition (Piecewise Continuous)

The function f is **piecewise continuous** on the interval $[a, b]$ if there is a partition $[x_0, x_1], [x_1, x_2], \cdots, [x_{n-1}, x_n]$ of $[a, b]$ such that

1. f is continuous on each open subinterval (x_{k-1}, x_k).
2. The left-handed and right-handed limits $\lim_{x \to x_k^{\pm}} f(x)$ exist and are finite (except possibly $\lim_{x \to a^-} f(x)$ and $\lim_{x \to b^+} f(x)$).

Theorem (Piecewise continuous implies integrable)

The function f is integrable on the interval $[a, b]$ if it is piecewise continuous on $[a, b]$.

Corollary (Continuous implies integrable)

If f is continuous on $[a, b]$ then it is integrable on $[a, b]$.

Definite integrals as limits of regular Riemann Sums

Example

Compute the following integrals as limits of regular Riemann Sums:

$$\begin{aligned}
\int_0^1 x^3 \, \mathrm{d}x &= \lim_{n \to \infty} R_n \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} f\left(a + k \cdot \tfrac{b-a}{n}\right) \tfrac{b-a}{n} \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} f\left(0 + k \cdot \tfrac{1-0}{n}\right) \tfrac{1-0}{n} \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} \left(\tfrac{k}{n}\right)^3 \tfrac{1}{n} \\
&= \lim_{n \to \infty} \sum_{k=1}^{n} \tfrac{k^3}{n^4} \\
&= \lim_{n \to \infty} \tfrac{1}{n^4} \sum_{i=1}^{n} k^3
\end{aligned}$$

Example (*continued*)

$$
\begin{aligned}
&= \lim_{n\to\infty} \tfrac{1}{n^4} \cdot \tfrac{n^2(n+1)^2}{4} \\
&= \lim_{n\to\infty} \tfrac{(n+1)^2}{4n^2} \\
&= \lim_{n\to\infty} \tfrac{n^2+2n+1}{4n^2} \\
&= \lim_{n\to\infty} \tfrac{1}{4} + \tfrac{2}{4n} + \tfrac{1}{4n^2} \\
&= \tfrac{1}{4}
\end{aligned}
$$

Example

Compute the following integrals as limits of regular Riemann Sums:

$$
\begin{aligned}
\int_0^b e^x \, \mathrm{d}x &= \lim_{n\to\infty} L_n \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(a + (k-1)\cdot \tfrac{b-a}{n}\right) \tfrac{b-a}{n} \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(0 + (k-1)\cdot \tfrac{b-0}{n}\right) \tfrac{b-0}{n} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} e^{\left(ib/n\right)} \tfrac{b}{n} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(e^{\left(b/n\right)}\right)^i \tfrac{b}{n} \\
&= \lim_{n\to\infty} \tfrac{b}{n} \cdot \left(\frac{1-\left(e^{\left(b/n\right)}\right)^n}{1-e^{b/n}}\right)
\end{aligned}
$$

Example (*continued*)

$$
\begin{aligned}
&= \lim_{n\to\infty} \tfrac{b}{n} \cdot \left(\frac{1-e^b}{1-e^{b/n}} \right) \\
&= b(1-e^b) \lim_{n\to\infty} \frac{n^{-1}}{1-e^{b/n}} \\
&= b(1-e^b) \lim_{n\to\infty} \frac{-n^{-2}}{e^{b/n} b n^{-2}} \\
&= b(1-e^b) \lim_{n\to\infty} \frac{-1}{e^{b/n} b} \\
&= b(1-e^b) \cdot \frac{-1}{e^0 b} \\
&= e^b - 1
\end{aligned}
$$

Properties of the definite integral

Theorem

If f and g are integrable and $a, b, c \in \mathbf{R}$ then

1. $\int_a^a f(x)\,\mathrm{d}x = 0$
2. $\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$
3. $\int_a^b \big(f(x) + g(x)\big)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$
4. $\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$
5. $\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$

Theorem

1. *If f is an integrable* **even** *function then*

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x.$$

2. *If f is an integrable* **odd** *function then*

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0.$$

Example

1. Suppose $\int_1^5 f(x)\,\mathrm{d}x = -7$ and $\int_1^7 f(x)\,\mathrm{d}x = 2$. Compute $\int_7^5 f(x)\,\mathrm{d}x$.

$$\begin{aligned}\int_7^5 f(x)\,\mathrm{d}x &= \int_7^1 f(x)\,\mathrm{d}x + \int_1^5 f(x)\,\mathrm{d}x \\ &= -\int_1^7 f(x)\,\mathrm{d}x + \int_1^5 f(x)\,\mathrm{d}x \\ &= -2 + (-7) \\ &= -9\end{aligned}$$

Computing integrals using area formulas or symmetry

Example

- $\int_0^6 \frac{x}{3}\,\mathrm{d}x = \frac{1}{2}bh = \frac{1}{2}(6)(2) = 6$
- $\int_0^{10} \frac{x}{5} + 9\,\mathrm{d}x = \frac{\ell_1+\ell_2}{2} \cdot b = \frac{9+11}{2} \cdot 10 = 100$
- $\int_{-3}^3 4x^3 + 6x\,\mathrm{d}x = 0$
- $\int_{-3}^3 \sin 8x\,\mathrm{d}x = 0$
- $\int_0^{4\pi} \cos x\,\mathrm{d}x = 0$
- $\int_{-4}^4 \sqrt{16-x^2}\,\mathrm{d}x = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi 4^2 = 8\pi$
- $\int_{-4}^0 \sqrt{16-x^2}\,\mathrm{d}x = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi 4^2 = 4\pi$

Example

1. $\int_5^0 3t - 4\sqrt{25 - t^2}\,\mathrm{d}t$

$$\begin{aligned}\int_5^0 3t - 4\sqrt{25 - t^2}\,\mathrm{d}t &= -\int_0^5 3t - 4\sqrt{25 - t^2}\,\mathrm{d}t \\ &= -\int_0^5 3t\,\mathrm{d}t + 4\int_0^5 \sqrt{25 - t^2}\,\mathrm{d}t \\ &= -\frac{1}{2}\cdot 5\cdot 15 + 4\cdot\frac{\pi 5^2}{4} \\ &= 25\pi - \frac{75}{2}\end{aligned}$$