Lecture 26 – October 26, 2018

Announcements

- Written Homework 5 due Tues. 10/30

Today

- Average value of a function
- Mean Value Theorem for integrals
- Substitution

Average value of a function

How should we define the average value of a function on an interval?

Example (Average value of a function)

1. What is the average value of the function

$$f(x) = 5$$

   on the interval $[0, 4]$?
   Answer: 5

2. What is the average value of the function

$$g(x) = \begin{cases} 2, & 0 \le x \le 2 \\ 6, & 2 < x \le 4 \end{cases}$$

   on the interval $[0, 4]$?
   Answer: $\frac{(2-0)2+(4-2)6}{4-0} = 4$

In general we can estimate the average value of the function $f(x)$ on the interval $[a, b]$ to be

$$\sum_{k=1}^{n} \frac{\Delta x}{b-a} f(x_k^*) = \frac{1}{b-a} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

Hence we expect that the average value of f on $[a, b]$ is

$$\lim_{n\to\infty} \frac{1}{b-a} \sum_{k=1}^{n} f(x_k^*) \Delta x = \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x$$

Definition (Average value of a function)

The **average value** of an integrable function f on the interval $[a, b]$ is

$$f_{\text{ave}} = \overline{f} = \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x$$

Example

Find the average value of the given functions on the given intervals:

1. $f(x) = \sin x$ on $[0, \pi]$
   $\overline{f} = \frac{\int_0^\pi \sin x \, \mathrm{d}x}{\pi - 0} = \frac{-\cos x|_0^\pi}{\pi} = \frac{-\cos\pi + \cos 0}{\pi} = \frac{2}{\pi}$
2. $g(x) = \frac{1}{1+x^2}$ on $[-1, 1]$
   $\overline{g} = \frac{\int_{-1}^{1} \frac{1}{1+x^2} \, \mathrm{d}x}{1-(-1)} = \frac{\arctan x|_{-1}^{1}}{2} = \frac{\arctan 1 - \arctan(-1)}{2} = \frac{\frac{\pi}{4} + \frac{\pi}{4}}{2} = \frac{\pi}{4}$
3. $h(x) = \frac{x+1}{x}$ on $[1, b]$
   $\overline{h} = \frac{\int_1^b \frac{x+1}{x} \, \mathrm{d}x}{b-1} = \frac{\int_1^b 1 + x^{-1} \, \mathrm{d}x}{b-1} = \frac{x + \ln|x||_1^b}{b-1} = \frac{b + \ln|b| - 1 + \ln 1}{2} = \frac{b + \ln b - 1}{b-1}$
4. As b approaches ∞ what happens to the average value of h on the interval $[1, b]$?
   $\lim_{b\to\infty} \overline{h} = \lim_{b\to\infty} \frac{b + \ln b - 1}{b-1} = \lim_{b\to\infty} \frac{1 + \frac{1}{b}}{1} = 1$

Theorem (Mean Value Theorem for Integrals)

If f is continuous on the interval $[a, b]$ then there is $c \in (a, b)$ such that

$$f(c) = \overline{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Relation to MVT

- Set $F(x) = \int_a^x f(t)\,dt$
- What does standard MVT say about F?
- There is $c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

- $F(a) = \int_a^x f(t)\,dt = 0$ and $F(b) = \int_a^b f(t)\,dt$ and by FTOC I $F'(c) = f(c)$.
- Thus equation above becomes

$$f(c) = \frac{\int_a^b f(t)\,dt - 0}{b - a} = \overline{f}$$

Example

For each of the following functions and intervals find the point (or points) predicted to exist by the Mean Value Theorem for Integrals.

1. $f(x) = \frac{3}{x}$ on $[-e, -1]$
   $\overline{f} = \frac{\int_{-e}^{-1} \frac{3}{x}\,dx}{-1-(-e)} = \frac{\ln|x|\,|_{-e}^{-1}}{e-1} = \frac{\ln 1 - \ln e}{e-1} = \frac{-1}{e-1}$
   $\frac{3}{c} = \frac{-1}{e-1}$ so $c = 3 - 3e$
2. $g(x) = x\sqrt{x^2 + 3x^4 + 20}$ on $[-10, 10]$
   odd function so $\overline{g} = 0$. $c = 0$

Recall from Wed:

Evaluating definite integrals

We now have 3 ways of evaluating definite integrals:

1. FTOC II (this is what we will use most of the time)
2. As a limit of Riemann Sums (see Lecture 24)
3. Using symmetry or area formulas that we already know

- Using FTOC II requires **antidifferentiation** techniques.
- Many of these will come from differentiation rules.
- Today we look at substitution (reverse of chain rule)

Substitution

Substitution

Recall Chain rule says:

$$\frac{\mathrm{d}}{\mathrm{dx}} F(g(x)) = F'(g(x)) \cdot g'(x)$$

This gives us the antidifferentiation rule:

$$F(g(x)) + C = \int F'(g(x)) \cdot g'(x) \, \mathrm{d}x$$

Applying substitution to an integral

In a successful substitution:

1. Choose a function $g(x)$ and try the substitution $u = g(x)$.
2. Must be able to find $g'(x) \, \mathrm{d}x$ in your integral to replace with $\mathrm{d}u$

Substitution

Example

$\int x^2 \sin x^3 \, \mathrm{d}x$

$$u = x^3, \mathrm{d}u = 3x^2 \, \mathrm{d}x$$

$$\begin{aligned}\int x^2 \sin x^3 \, \mathrm{d}x &= \int \tfrac{1}{3} \sin x^3 3x^2 \, \mathrm{d}x \\ &= \int \tfrac{1}{3} \sin u \, \mathrm{d}u \\ &= -\tfrac{1}{3} \cos u + C \\ &= -\tfrac{1}{3} \cos x^3 + C\end{aligned}$$

CHECK

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\left(-\tfrac{1}{3}\cos x^3 + C\right) = \tfrac{1}{3}(\sin x^3) \cdot 3x^2 = x^2 \sin x^3 \checkmark$$

Success vs failure in substitution

In a successful substitution:

1. $\mathrm{d}u = g'(x)\, \mathrm{d}x$ **must** be a factor of integrand (up to multiplication by a constant)
2. After substituting in u there should be no x or $\mathrm{d}x$ in the integrand.
3. In a substitution problem there should never be a moment when there are both x and u in the integrand. (major points off even if you get the right answer eventually)
4. Don't be afraid to fail! (Look for possible u's **and** $\mathrm{d}u$'s)
5. As a last resort solve for x in terms of u

Example

$\int \frac{x}{1+x^4}\,\mathrm{d}x$
Possible u's:

$$u = x^4, \quad u = 1 + x^4, \quad u = x, \quad u = \frac{1}{1+x^4}$$

Possible du's

$$\mathrm{d}u = cx\,\mathrm{d}x, \quad \mathrm{d}u = \frac{c\,\mathrm{d}x}{1+x^4}$$

Correct choice: $u = x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$

$$\begin{aligned}\int \frac{x}{1+x^4}\,\mathrm{d}x &= \int \frac{1}{1+(x^2)^2} \cdot \tfrac{1}{2} \cdot 2x\,\mathrm{d}x \\ &= \int \frac{1}{1+u^2} \cdot \tfrac{1}{2}\,\mathrm{d}u \\ &= \tfrac{1}{2}\arctan u + C \\ &= \tfrac{1}{2}\arctan(x^2) + C\end{aligned}$$

CHECK

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{2}\arctan x^2 + C\right) = \frac{1}{2} \cdot \frac{2x}{1+(x^2)^2} = \frac{x}{1+x^4} \checkmark$$

Example (More basic substitution problems)

1. $\int \cos^7 x \sin x\,\mathrm{d}x$

$$u = \cos x, \mathrm{d}u = -\sin x\,\mathrm{d}x$$

$$\begin{aligned}\int \cos^7 x \sin x\,\mathrm{d}x &= \int -u^7\,\mathrm{d}u \\ &= -\tfrac{u^8}{8} + C \\ &= -\tfrac{\cos^8 x}{8} + C\end{aligned}$$

2. $\int \tan x \sec^5 x\,\mathrm{d}x$

$$u = \sec x, \mathrm{d}u = \tan x \sec x\,\mathrm{d}x$$

$$\begin{aligned}\int \tan x \sec^5 x\,\mathrm{d}x &= \int u^4\,\mathrm{d}u \\ &= \tfrac{u^5}{5} + C \\ &= \tfrac{\sec^5 x}{5} + C\end{aligned}$$

Example (More basic substitution problems)

3. $\int \frac{e^x\,\mathrm{d}x}{\sqrt{1-e^{2x}}}$

$$u = e^x, \mathrm{d}u = e^x\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{e^x\,\mathrm{d}x}{\sqrt{1-e^{2x}}} &= \int \frac{\mathrm{d}u}{\sqrt{1-u^2}} \\ &= \arcsin u + C \\ &= \arcsin e^x + C\end{aligned}$$

4. $\int \frac{\mathrm{d}x}{(2x-5)^6}$

$$u = 2x-5, \mathrm{d}u = 2\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{\mathrm{d}x}{(2x-5)^6} &= \int \tfrac{1}{2} u^{-6}\,\mathrm{d}u \\ &= \tfrac{u^{-5}}{2(-5)} + C \\ &= -\tfrac{1}{10(2x-5)^5} + C\end{aligned}$$

Substitution with definite integrals

Theorem (Substitution in a definite integral)

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

Example (Definite integral involving a substitution)

$\int_2^3 x^2 \sin x^3\,\mathrm{d}x$

$$u = x^3, \mathrm{d}u = 3x^2\,\mathrm{d}x$$

$$\begin{aligned}\int_2^3 x^2 \sin x^3\,\mathrm{d}x &= \int_2^3 \tfrac{1}{3} \sin x^3 3x^2\,\mathrm{d}x \\ &= \int_{2^3}^{3^3} \tfrac{1}{3} \sin u\,\mathrm{d}u = -\tfrac{1}{3} \cos u\Big|_{2^3}^{3^3}\end{aligned}$$

Example (Evaluate the following definite integrals)

1. $\int_0^{2\pi} \frac{\cos x}{3+\sin x}\,\mathrm{d}x$

$$u = 3 + \sin x, \mathrm{d}u = \cos x\,\mathrm{d}x$$

$$\begin{aligned}\int_0^{2\pi} \frac{\cos x}{3+\sin x}\,\mathrm{d}x &= \int_{3+\sin 0}^{3+\sin 2\pi} \frac{\mathrm{d}u}{u} \\ &= \ln|u|\Big|_{u=3}^{3} \\ &= \ln|3| - \ln|3| \\ &= 0\end{aligned}$$

2. $\int_0^1 \frac{e^{2t}}{1+e^{2t}}\,\mathrm{d}t$

$$u = 1 + e^{2t}, \mathrm{d}u = 2e^{2t}\,\mathrm{d}t$$

$$\begin{aligned}\int_0^1 \frac{e^{2t}}{1+e^{2t}}\,\mathrm{d}t &= \int_{1+e^{2\cdot 0}}^{1+e^{2\cdot 1}} \frac{\mathrm{d}u}{2u} \\ &= \ln|u|\Big|_{u=2}^{1+e^2} \\ &= \ln|1+e^2| - \ln|2|\end{aligned}$$

Example (Evaluate the following definite integrals)

3. $\int_0^1 \frac{e^t}{1+e^{2t}}\,\mathrm{d}t$

$$u = e^t, \mathrm{d}u = e^t\,\mathrm{d}t$$

$$\begin{aligned}\int_0^1 \frac{e^t}{1+e^{2t}}\,\mathrm{d}t &= \int_{e^0}^{e^1} \frac{\mathrm{d}u}{1+u^2} \\ &= \arctan u\Big|_{u=1}^{e} \\ &= \arctan e - \arctan 1 \\ &= \arctan e - \frac{\pi}{4}\end{aligned}$$

Linear substitutions

The substitution $u = mx + b$ can always be made to work

Example

1. $\int e^{3x+2}\,\mathrm{d}x$
2. $\int (x-5)\sqrt{2-x}\,\mathrm{d}x$
3. $\int \frac{\mathrm{d}x}{\sqrt{25-x^2}}$
4. $\int \frac{\mathrm{d}x}{\sqrt{16-x^2}} = \int \frac{1}{4} \cdot \frac{\mathrm{d}x}{\sqrt{1-(x/4)^2}}$, $u = x/4$, $\mathrm{d}u = 1/4\,\mathrm{d}x$

Example

Find the following indefinite integrals

1. $\int \frac{\ln(y^3)\,\mathrm{d}y}{y}$
2. $\int \frac{\mathrm{d}t}{t^2+1} = \arctan t + C$
3. $\int \frac{t\,\mathrm{d}t}{t^2+1}$, $u = t^2+1$, $\mathrm{d}u = 2t\,\mathrm{d}t$,
   $\int \frac{\mathrm{d}u}{2u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|t^2+1| + C$
4. $\int \frac{t^2\,\mathrm{d}t}{t^2+1}$, $\int 1 - \frac{1}{t^2+1}\,\mathrm{d}t = t - \arctan t + C$
5. $\int (6x^2 + 10 - 2e^x)\sin(x^3 + 5x - e^x)\,\mathrm{d}x$
6. $\int xe^{(x^2)}\,\mathrm{d}x$ NOTE: $u = x^2$ AND $u = e^{(x^2)}$ work

Forcing a substitution to work

You can do the substitution $u = g(x)$ if the function g has an inverse.

1. $x = g^{-1}(u)$
2. $\mathrm{d}x = (g^{-1})'(u)\,\mathrm{d}u$

Example

Substitute $u = \ln x$ in the integral $\int x \arctan(3x)\,\mathrm{d}x$
$u = \ln x$ so $x = e^u$ and $\mathrm{d}x = e^u\,\mathrm{d}u$

$$\int x \arctan(3x)\,\mathrm{d}x = \int e^u \arctan(3e^u) e^u\,\mathrm{d}u$$

Of course this was't actually a help

Example

Substitute $u = \sqrt{5+x}$ in the integral $\int \frac{\mathrm{d}x}{\sqrt[3]{4+\sqrt{5+x}}}$ then compute the indefinite integral
$u = \sqrt{5+x}$ so $x = u^2 - 5$ and $\mathrm{d}x = 2u\,\mathrm{d}u$

$$\int \frac{\mathrm{d}x}{\sqrt[3]{4+\sqrt{5+x}}} = \int \frac{2u\,\mathrm{d}u}{\sqrt[3]{4+u}}$$

$v = 4 + u$ so $u = v - 4$ and $\mathrm{d}v = \mathrm{d}u$

$$\begin{aligned}\int \frac{2u\,\mathrm{d}u}{\sqrt[3]{4+u}} &= \int \frac{2(v-4)\,\mathrm{d}v}{\sqrt[3]{v}} \\ &= \int 2v^{2/3} - 8v^{-1/3}\,\mathrm{d}v \\ &= \tfrac{6}{5}v^{5/3} - 12v^{2/3} + C\end{aligned}$$

Example

$$= \tfrac{6}{5}v^{5/3} - 12v^{2/3} + C$$
$$= \tfrac{6}{5}(4+u)^{5/3} - 12(4+u)^{2/3} + C$$
$$= \tfrac{6}{5}(4+\sqrt{5+x})^{5/3} - 12(4+\sqrt{5+x})^{2/3} + C$$

Example

Substitute $u = 3 + \sqrt{x}$ in the integral $\int \frac{x\,\mathrm{d}x}{\sqrt{3+\sqrt{x}}}$ then compute the indefinite integral
$u = 3 + \sqrt{x}$ so $(u-3)^2 = x$ and $\mathrm{d}x = 2(u-3)\,\mathrm{d}u$

$$\int \frac{x\,\mathrm{d}x}{\sqrt{3+\sqrt{x}}} = \int \frac{(u-3)^2 2(u-3)\,\mathrm{d}u}{\sqrt{u}}$$
$$= 2\int u^{-1/2}(u-3)^3\,\mathrm{d}u$$
$$= 2\int u^{-1/2}(u^3 - 3u^2 + 9u - 27)\,\mathrm{d}u$$
$$= 2\int u^{(5/2)} - 3u^{(3/2)} + 9u^{(1/2)} - 27u^{-1/2}\,\mathrm{d}u$$
$$= 2\cdot\tfrac{2}{7}u^{(7/2)} - 6\cdot\tfrac{2}{5}u^{(5/2)} + 18\cdot\tfrac{2}{3}u^{(3/2)} - 54\cdot 2u^{1/2} + C$$
$$= \tfrac{4}{7}(3+\sqrt{x})^{(7/2)} - \tfrac{12}{5}(3+\sqrt{x})^{(5/2)} + 12(3+\sqrt{x})^{(3/2)} - 108(3+\sqrt{x})^{1/2} + C$$