Lecture 28 – October 31, 2018

Announcements

- Office hours Mon. & Wed. 12:30pm-2pm in Math Tower (MW) room 650

Today

- Volume by slicing

Volume by slicing

Volume by slicing

If S is a 3-dimensional solid shape in xyz-space contained entirely between the planes $x = a$ and $x = b$ such that the intersection of the plane $x = c$ has area $A(c)$ for all $c \in [a, b]$ then the volume of S is

$$\text{Volume} = \int_a^b A(x)\,\mathrm{d}x$$

Volumes of rotation by slicing

If R is the 2-dimensional region between the curves $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$ then the volume of the 3-dimensional object one gets by rotating R about the line $y = c$ is

$$\text{Volume} = \left| \int_a^b \pi(f(x) - c)^2 - \pi(g(x) - c)^2\,\mathrm{d}x \right|$$

Volumes of rotation by slicing

If R is the 2-dimensional region between the curves $x = f(y)$ and $x = g(y)$ between $y = c$ and $y = d$ then the volume of the 3-dimensional object one gets by rotating R about the line $x = k$ is

$$\text{Volume} = \left| \int_c^d \pi(f(y) - k)^2 - \pi(g(y) - k)^2 \, \mathrm{d}y \right|$$

Example (Volume of a cone)

Show that the volume formula for a circular cone with height h and radius r is correct.

- Line through $(0, 0)$ and (h, r) is $y = \frac{r}{h}(x - 0) + 0$
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$$\begin{aligned}
\text{Volume} &= \int_0^h \pi\left(\tfrac{rx}{h} - 0\right)^2 \mathrm{d}x \\
&= \int_0^h \pi\left(\tfrac{r^2}{h^2}\right)x^2 \, \mathrm{d}x \\
&= \pi\left(\tfrac{r^2}{h^2}\right) \int_0^h x^2 \, \mathrm{d}x \\
&= \pi\left(\tfrac{r^2}{h^2}\right) \tfrac{x^3}{3}\Big|_{x=0}^{h} \\
&= \pi\left(\tfrac{r^2}{h^2}\right)\tfrac{h^3}{3} - \pi\left(\tfrac{r^2}{h^2}\right)\tfrac{0^3}{3} \\
&= \tfrac{\pi}{3} r^2 h
\end{aligned}$$

Example (Volume of a sphere)

Show that the volume formula for a sphere of radius r is correct.

- Sphere is volume of rotation when bounded region between $y = \sqrt{r^2 - x^2}$ and $y = 0$ is rotated about x-axis
- $$\begin{aligned} \text{Volume} &= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 \mathrm{d}x \\ &= \int_{-r}^{r} \pi r^2 - \pi x^2 \, \mathrm{d}x \\ &= \pi r^2 x - \tfrac{\pi x^3}{3}\Big|_{x=-r}^{r} \\ &= \pi r^2 r - \tfrac{\pi r^3}{3} - \left(\pi r^2(-r) - \tfrac{\pi(-r)^3}{3}\right) \\ &= \tfrac{4\pi r^3}{3} \end{aligned}$$

4-Volume of a 4-sphere (will not be on any exams)

Give an integral to compute 4-Volume of a 4-sphere

- 4-sphere has spherical "cross-sections" with radius $y = \sqrt{r^2 - x^2}$
- $$\text{4-Volume} = \int_{-r}^{r} \tfrac{4\pi}{3} \left(\sqrt{r^2 - x^2}\right)^3 \mathrm{d}x$$
- This is a tough integral to evaluate directly right now
- In Section 7.4 we will see that substituting $x = r \sin u$ might be helpful.

Example

Set up integral for when the **bounded region** R between the curves

$$y = x^2, \qquad x = y^3$$

is rotated about

1. the line $y = -3$
2. the line $x = 4$.

Part 1: Rotating R about $y = -3$

- GRAPH ALL CURVES
- For curve $y = x^2$ we already have y as function of x.
- For curve $x = y^3$ we must solve for y as function(s) of x

$$x = y^3$$
$$y = x^{1/3}$$

- Thus R is bounded region between $y = f(x) = x^2$ and $y = g(x) = x^{1/3}$

Example (*continued*)

- x-coordinates of curve intersections:

$$f(x) = g(x)$$
$$x^2 = x^{1/3}$$
$$x^6 = x$$
$$x^6 - x = 0$$
$$x(x^5 - 1) = 0$$

  So $x = 0$ or $x^5 = 1$ hence $x = 1$.
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$$\text{Volume} = \int_0^1 \pi(x^{1/3} - (-3))^2 - \pi(x^2 - (-3))^2 \, dx$$

Example (*continued*)

Part 2: Rotating R about $x = 4$

- REGRAPH ALL CURVES
- For curve $x = y^3$ we already have x as function of y.
- For curve $y = x^2$ we must solve for y as function(s) of x

$$x = y^2$$
$$x = \pm\sqrt{y}$$

- Thus R is bounded region between $x = h(y) = y^3$ and $x = j(y) = \sqrt{y}$. (From graph observe that $x = \sqrt{y}$ bounds region we are studying, not $x = -\sqrt{y}$)

Example (*continued*)

- y-coordinates of curve intersections:

$$h(y) = j(y)$$
$$y^3 = \sqrt{y}$$
$$y^6 = y$$
$$y^6 - y = 0$$
$$y(y^5 - 1) = 0$$

  So $y = 0$ or $y^5 = 1$ hence $y = 1$.

- $$\text{Volume} = \int_0^1 \pi(y^3 - 4)^2 - \pi(\sqrt{y} - 4)^2 \, dy$$

- Note: the volumes computed in Part 1 and Part 2 will almost certainly differ since they are volumes of different 3-dimensional objects.

Example

Let T be the 2-dimensional triangular region in xyz-space with vertices $(0,0,0)$, $(1,0,0)$ and $(0,2,0)$ all of which are in the xy-plane. Create a 3-dimensional shape P whose cross sectional intersection with the plane $x = c$ is a square whose its bottom side is the intersection of the plane $x = c$ with the triangle T. Set up an integral for the volume of P.

- In xy-plane equation for slanted side of T is

$$y = \tfrac{0-2}{1-0}(x-0) + 2 = -2x + 2$$

- Slice of P for x is square with side length $-2x + 2$ and thus area $(-2x+2)^2$
- Volume of P is therefore

$$\text{Volume} = \int_0^1 (-2x+2)^2 \, \mathrm{d}x$$

Example (Volume of intersection of two cylinders)

Let C_1 be the solid circular cylinder of radius 1 with axis the x-axis in xyz-space and let C_2 be the solid circular cylinder of radius 1 with axis the y-axis . Set up an integral to compute the volume of the **intersection** of cylinders C_1 and C_2.

- We will slice along planes parallel to the xy-plane (that is, along planes with equation $z = c$).
- Intersection of C_1 with plane at height z is strip of width w.
- Intersection of C_2 with plane at height z is strip of same width w.
- Thus intersection of intersection of C_1 and C_2 with plane at height z is $w \times w$ **square** with area $A(z) = w^2$
- Viewing cylinder C_1 end-on we see that $z^2 + (w/2)^2 = 1^2$. Thus

$$(w/2)^2 = 1 - z^2$$
$$w = 2\sqrt{1-z^2}$$

- Volume of intersection of C_1 and C_2 is

$$\text{Volume} = \int_{-1}^{1} \left(2\sqrt{1-z^2}\right)^2 \mathrm{d}z$$