Lecture 29 – November 2, 2018

Today

- Volume by cylindrical shells

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If R is the 2-dimensional region between the curves $y = f(x)$ and $y = g(x)$ and between $x = a$ and $x = b$ then the volume of the 3-dimensional object one gets by rotating R about the line $x = c$ is

$$\text{Volume} = \int_a^b 2\pi|x - c| \cdot |f(x) - g(x)| \, dx$$

Volume by cylindrical shells

If R is the 2-dimensional region between the curves $x = f(y)$ and $x = g(y)$ and between $y = c$ and $y = d$ then the volume of the 3-dimensional object one gets by rotating R about the line $y = k$ is

$$\text{Volume} = \int_c^d 2\pi|y - k| \cdot |f(y) - g(y)| \, dy$$

Example (Volume of a cone)

Show that the volume formula for a circular cone with height h and radius r is correct.

- Line through $(0,0)$ and (h,r) is $y = \frac{r}{h}(x-0)+0$ or $x = \frac{hy}{r}$
-

$$\begin{aligned}
\text{Volume} &= \int_0^r 2\pi|y-0| \cdot |h - \tfrac{hy}{r}|\,\mathrm{d}y \\
&= 2\pi h \int_0^r (y - \tfrac{y^2}{r})\,\mathrm{d}y \\
&= 2\pi h\,(\tfrac{y^2}{2} - \tfrac{y^3}{3r})\Big|_{y=0}^{r} \\
&= 2\pi h(\tfrac{r^2}{2} - \tfrac{r^3}{3r}) - 2\pi h(\tfrac{0^2}{2} - \tfrac{0^3}{3r}) \\
&= \tfrac{\pi}{3} r^2 h
\end{aligned}$$

Example (Volume of a sphere)

Show that the volume formula for a sphere of radius r is correct.

- Sphere is volume of rotation when bounded region between $y = \sqrt{r^2 - x^2}$ and $y = 0$ is rotated about x-axis
- Solving for x we get $x = \pm\sqrt{r^2 - y^2}$.
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$$\begin{aligned}
\text{Volume} &= \int_0^r 2\pi(y-0)\left(\sqrt{r^2-y^2} - (-\sqrt{r^2-y^2})\right)\mathrm{d}y \\
&= \int_0^r 4\pi y\sqrt{r^2-y^2}\,\mathrm{d}y \\
u &= r^2 - y^2, \qquad \mathrm{d}u = -2y\,\mathrm{d}y \\
&= \int_{r^2-0^2}^{r^2-r^2} -2\pi\sqrt{u}\,\mathrm{d}u \\
&= -2\pi \cdot \tfrac{2}{3} u^{3/2}\Big|_{u=r^2}^{0} \\
&= -2\pi \cdot \tfrac{2}{3} \cdot 0^{3/2} + 2\pi \cdot \tfrac{2}{3}(r^2)^{3/2} \\
&= \tfrac{4\pi r^3}{3}
\end{aligned}$$

Example

Set up integral for volume when the **bounded region** R between the curves

$$y = x^2, \qquad x = y^3$$

is rotated about

1. the line $y = -3$
2. the line $x = 4$.

Part 1: Rotating R about $y = -3$

- GRAPH ALL CURVES
- For curve $x = y^3$ we already have x as function of y.
- For curve $y = x^2$ we must solve for y as function(s) of x

$$x = y^2$$
$$x = \pm\sqrt{y}$$

- Thus R is bounded region between $x = h(y) = y^3$ and $x = j(y) = \sqrt{y}$. (From graph observe that $x = \sqrt{y}$ bounds region we are studying, not $x = -\sqrt{y}$)

Example (*continued*)

- y-coordinates of curve intersections:

$$h(y) = j(y)$$
$$y^3 = y^{1/2}$$
$$y^6 = y$$
$$y^6 - y = 0$$
$$y(y^5 - 1) = 0$$

So $y = 0$ or $y^5 = 1$ hence $y = 1$.

- $$\text{Volume} = \int_0^1 2\pi(y - (-3))(\sqrt{y} - y^3)\,dy$$

Example (*continued*)

Part 2: Rotating R about $x = 4$

- REGRAPH ALL CURVES
- For curve $y = x^2$ we already have y as function of x.
- For curve $x = y^3$ we must solve for y as function(s) of x

$$x = y^3$$
$$y = x^{1/3}$$

- Thus R is bounded region between $y = f(x) = x^2$ and $y = g(x) = x^{1/3}$

Example (*continued*)

- y-coordinates of curve intersections:

$$f(x) = g(x)$$
$$x^2 = x^{1/3}$$
$$x^6 = x$$
$$x^6 - x = 0$$
$$x(x^5 - 1) = 0$$

  So $x = 0$ or $x^5 = 1$ hence $x = 1$.
- $$\text{Volume} = \int_0^1 2\pi(4 - x)(x^{1/3} - x^2)\,dx$$
- Note: the volumes computed in Part 1 and Part 2 will almost certainly differ since they are volumes of different 3-dimensional objects.
- However the integrals about should agree with the values of the corresponding integrals from Wednesday's lecture.

Example

Give and integral for the volume when the bounded region below the line $y = x + 2$ and below the curve $y = 3 - e^x$ but above the x-axis is rotated about the line $x = 7$.

- We need the intersections of the curves $y = x + 2$, $y = 3 - e^x$ and $y = 0$.

$$\begin{aligned} x + 2 &= 3 - e^x \\ x + 1 &= e^x \\ x &= 0 \end{aligned} \qquad \begin{aligned} x + 2 &= 0 \\ x &= -2 \end{aligned} \qquad \begin{aligned} 3 - e^x &= 0 \\ x &= \ln 3 \end{aligned}$$

- Volume is therefore

$$\text{Volume} = \int_{-2}^{0} 2\pi(7 - x)(x + 2)\,\mathrm{d}x + \int_{0}^{\ln 3} 2\pi(7 - x)(3 - e^x)\,\mathrm{d}x$$