Lecture 31 – November 7, 2018

Announcements

- Office hours Mon. & Wed. 12:30pm-2pm in Math Tower (MW) room 650
- Written Homework 6 Due Tues. 11/13
- Midterm 3 Thursday, 11/15 6-6:55pm in Journalism Building (JR) 300
- Practice Midterm 3 and Study Guide 3 Posted

Today

- Surfaces of revolution

Frustrum of a cone

- From washers we know area when line segment perpendicular to axis of rotation is rotated about that axis is

$$\pi(R^2 - r^2)$$

- From cylindrical shells we know area when line segment parallel to axis of rotation is is rotated about that axis is

$$2\pi rh$$

- Today we start by **interpolating** between these extremes by computing area when arbitrary line segment is rotated about an axis yielding the **frustrum of a cone**

- Let ℓ be distance from tip of cone to point on circle at base.

$$\ell = \sqrt{r^2 + h^2}$$

- Area of sides of cone with height h and base radius r is

$$\pi \ell^2 \cdot \frac{2\pi r}{2\pi \ell} = \pi r \sqrt{r^2 + h^2}$$

- Area of frustrum with smaller radius r_1 larger radius r_2 and height h is difference between area of two cones:
- Let H be height of small cone. By similar triangles

$$\begin{aligned}
\frac{H}{r_1} &= \frac{H+h}{r_2} \\
Hr_2 &= Hr_1 + hr_1 \\
Hr_2 - Hr_1 &= hr_1 \\
H &= \frac{hr_1}{r_2 - r_1}
\end{aligned}$$

- Thus area of frustrum is

$$\begin{aligned}
\text{Area of frustrum} &= (\text{Area of large cone}) - (\text{Area of small cone}) \\
&= \pi r_2 \sqrt{r_2^2 + (H+h)^2} - \pi r_1 \sqrt{r_1^2 + H^2} \\
&= \pi r_2 \sqrt{r_2^2 + \left(\tfrac{hr_1}{r_2 - r_1} + h\right)^2} - \pi r_1 \sqrt{r_1^2 + \left(\tfrac{hr_1}{r_2 - r_1}\right)^2} \\
&= \pi r_2 \sqrt{\tfrac{r_2^2 (r_2 - r_1)^2}{(r_2 - r_1)^2} + \left(\tfrac{hr_1}{r_2 - r_1} + \tfrac{hr_2 - hr_1}{r_2 - r_1}\right)^2} \\
&\qquad - \pi r_1 \sqrt{\tfrac{r_1^2 (r_2 - r_1)^2}{(r_2 - r_1)^2} + \left(\tfrac{hr_1}{r_2 - r_1}\right)^2} \\
&= \pi r_2 \sqrt{\tfrac{r_2^2 (r_2 - r_1)^2 + h^2 r_2^2}{(r_2 - r_1)^2}} - \pi r_1 \sqrt{\tfrac{r_1^2 (r_2 - r_1)^2 + h^2 r_1^2}{(r_2 - r_1)^2}} \\
&= \frac{\pi r_2^2 \sqrt{(r_2 - r_1)^2 + h^2} - \pi r_1^2 \sqrt{(r_2 - r_1)^2 + h^2}}{r_2 - r_1} \\
&= \pi (r_2 + r_1) \sqrt{(r_2 - r_1)^2 + h^2}
\end{aligned}$$

Area of frustrum

Area of a frustrum with top radius r_1 bottom radius r_2 and height h is

$$\text{Area of frustrum} = \pi(r_2 + r_1)\sqrt{(r_2 - r_1)^2 + h^2}$$

- Does this area formula interpolate between area of a cylindrical shell $2\pi rh$ and area of an annulus $\pi R^2 - \pi r^2$?
- For a cylindrical shell of radius r we want $r_1 = r_2 = r$ and $h = h$

$$\pi(r_2 + r_1)\sqrt{(r_2 - r_1)^2 + h^2} = \pi(r + r)\sqrt{(r - r)^2 + h^2} = 2\pi rh$$

- For an annulus with inner radius r and outer radius R we want $r_1 = r$, $r_2 = R$ and $h = 0$

$$\pi(r_2 + r_1)\sqrt{(r_2 - r_1)^2 + h^2} = \pi(R + r)\sqrt{(R - r)^2 + 0^2} = \pi(R^2 - r^2)$$

- Thus our formula for area of frustrum does interpolate well between area of a cylindrical shell and area of an annulus.

Estimate of surface area of surface of revolution

Estimate of area when curve $y = f(x)$ between $x = a$ and $x = b$ is rotated about x-axis is

$$\begin{aligned}\text{Surface Area} &\approx \sum_{k=1}^{n} \pi(|f(x_k)| + |f(x_{k-1})|)\sqrt{\Delta y_k^2 + \Delta x^2} \\ &= \sum_{k=1}^{n} \pi(|f(x_k)| + |f(x_{k-1})|)\left(\sqrt{\left(\frac{\Delta y_k}{\Delta x}\right)^2 + 1}\right)\Delta x\end{aligned}$$

Theorem

If f has continuous derivative on $[a, b]$ then the area of the surface of revolution when $y = f(x)$ between $x = a$ and $x = b$ is rotated about x-axis is

$$\textit{Surface Area} = \int_a^b 2\pi|f(x)|\sqrt{1 + (f'(x))^2}\,\mathrm{d}x$$

Proof Idea.

The exact length of the curve $y = f(x)$ between $x = a$ and $x = b$ is

$$\text{Surface Area} = \lim_{n\to\infty} \sum_{k=1}^{n} \pi(|f(x_k)| + |f(x_{k-1})|) \left(\sqrt{\left(\frac{\Delta y_k}{\Delta x}\right)^2 + 1} \right) \Delta x$$

Using MVT trick from arc length proof we can replace $\frac{\Delta y_k}{\Delta x}$ with $f'(x_k^*)$ for some point $x_k^* \in (x_{k-1}, x_k)$. Finally, we note that as Δx approaches 0 we have x_k and x_{k-1} approaching x_k^*. Thus the sums given above approach Riemann Sums for the function

$$2\pi |f(x)| \sqrt{1 + (f'(x))^2}$$

Thus

$$\text{Surface Area} = \int_a^b 2\pi |f(x)| \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

□

General surface areas of revolution

Surface area rotating about $y = c$

The surface area when the curve $y = f(x)$ between $x = a$ and $x = b$ is rotated about the line $y = c$ is

$$\text{Surface Area} = \int_a^b 2\pi |f(x) - c| \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

Surface area rotating about $x = k$

The surface area when the curve $x = g(y)$ between $y = c$ and $y = d$ is rotated about the line $x = k$ is

$$\text{Surface Area} = \int_c^d 2\pi |g(y) - k| \sqrt{1 + (g'(y))^2}\, \mathrm{d}y$$

Example (Surface Area of a sphere)

Find surface area of sphere of radius r

- EXPECTED ANSWER: Surface Area $= 4\pi r^2$
- Note: Our surface area theorem does not technically apply on enpoints of domain of integration $[-r, r]$. But answer we get will be correct.
- Find surface area when $f(x) = \sqrt{r^2 - x^2}$ between $x = -r$ and $x = r$ is rotated about x-axis.
- $f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{r^2-x^2}}$

$$\begin{aligned}
\text{Surface Area} &= \int_{-r}^{r} 2\pi \left|\sqrt{r^2 - x^2} - 0\right| \sqrt{1 + \left(\tfrac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx \\
&= \int_{-r}^{r} 2\pi \sqrt{r^2 - x^2} \sqrt{\tfrac{r^2}{r^2-x^2}}\, dx \\
&= \int_{-r}^{r} 2\pi r\, dx \\
&= 2\pi r x\Big|_{x=-r}^{r} \\
&= 4\pi r^2
\end{aligned}$$

Example

Compute surface area when $y = x^3$ between $x = 0$ and $x = 3$ is rotated about the x-axis.

- $f'(x) = 3x^2$

$$\begin{aligned}
\text{Surface area} &= \int_0^3 2\pi |x^3 - 0| \sqrt{1 + (3x^2)^2}\, dx \\
&= \int_0^3 2\pi x^3 \sqrt{1 + 9x^4}\, dx \\
u = 1 + 9x^4, &\qquad du = 36x^3\, dx \\
&= \int_{1+9\cdot 0^4}^{1+9\cdot 3^4} \pi\sqrt{u} \cdot \tfrac{1}{18}\, du \\
&= \tfrac{\pi}{18} \cdot \tfrac{2}{3} u^{3/2}\Big|_{u=1}^{730} \\
&= \tfrac{\pi}{27} \cdot 730^{3/2} - \tfrac{\pi}{27} \cdot 1^{3/2}
\end{aligned}$$

Example

Compute surface area when $y = x^2$ between $x = -4$ and $x = 0$ is rotated about the y-axis.

- $x = -\sqrt{y}$
- $x' = -\frac{1}{2}y^{-1/2}$

$$
\begin{aligned}
\text{Surface area} &= \int_{0^2}^{(-4)^2} 2\pi|-\sqrt{y}-0|\sqrt{1+(-\tfrac{1}{2}y^{-1/2})^2}\,\mathrm{d}y \\
&= \int_0^{16} 2\pi\sqrt{y}\sqrt{1+\tfrac{1}{4y}}\,\mathrm{d}y \\
&= \int_0^{16} 2\pi\sqrt{y+\tfrac{1}{4}}\,\mathrm{d}y \\
u &= y+\tfrac{1}{4}, \qquad \mathrm{d}u = \mathrm{d}y \\
&= \int_{0+1/4}^{16+1/4} 2\pi\sqrt{u}\,\mathrm{d}u \\
&= \tfrac{4\pi}{3}u^{3/2}\Big|_{u=1/4}^{65/4} \\
&= \tfrac{4\pi}{3}\cdot(65/4)^{3/2} - \tfrac{4\pi}{3}\cdot(1/4)^{3/2}
\end{aligned}
$$