Lecture 32 – November 9, 2018

Announcements

- Office hours Mon. & Wed. 12:30pm-2pm in Math Tower (MW) room 650
- Written Homework 6 Due Tues. 11/13
- Midterm 3 Thursday, 11/15 6-6:55pm in Journalism Building (JR) 300
- Practice Midterm 3 and Study Guide 3 Posted

Today

- Mass from 1-dimensional density
- Work

Mass from 1-dimensional density

Mass from 1-dimensional density

If $\rho(x)$ is the density (in (mass)/(unit length)) of a bar at position x between $x = a$ and $x = b$ then the total mass of the bar is

$$m = \int_a^b \rho(x)\, \mathrm{d}x$$

Example (Mass of bar with variable density)

Compute the total mass of a bar with density

$$\rho(x) = 10 - x^2 \,\mathrm{kg/m}$$

for a bar with one end at $x = 1$ and the other end at $x = 3$.

$$\begin{aligned}\text{Total mass of bar} &= \int_1^3 10 - x^2 \,\mathrm{d}x \\ &= 10x - \tfrac{x^3}{3}\Big|_{x=1}^3 \\ &= 10 \cdot 3 - \tfrac{3^3}{3} - 10 \cdot 1 + \tfrac{1^3}{3} = \tfrac{34}{3}\,\mathrm{kg}\end{aligned}$$

Work

Work with variable force

If $F(x)$ is the force applied at position x then the **work** or **energy** required to move from position $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,\mathrm{d}x$$

SI unit for work is $\mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}^2 = \mathrm{N} \cdot \mathrm{m} = \mathrm{J}$

Modeling springs

Hooke's Law is a model for springs and says that for each spring there is a **spring constant** k such that an application of a force of

$$F(x) = kx$$

will displace the end of the spring by a distance x.

Example

A $50\,\mathrm{N}$ force stretches a spring by $2\mathrm{m}$. What is the spring constant? How much energy is stored in the spring when it is stretched by this weight?

- Spring constant

$$\begin{aligned} 50 &= k \cdot 2 \\ k &= 25 \end{aligned}$$

- Energy stored in spring

$$\begin{aligned} W &= \int_0^2 F(x)\,\mathrm{d}x \\ &= \int_0^2 25x\,\mathrm{d}x \\ &= \tfrac{25x^2}{2}\Big|_{x=0}^{2} \\ &= \tfrac{25 \cdot 2^2}{2} - \tfrac{25 \cdot 0^2}{2} = 50\,\mathrm{J} \end{aligned}$$

Lifting Problems

Work to lift liquid

1. tank has cross-sectional area $A(y)$ at height y.
2. bottom of tank at $y = a$
3. top of liquid at height $y = b$
4. density of liquid in tank is ρ (mass)/(unit volume).
5. acceleration of gravity is g
6. height to lift water is h

Then the total work to lift all liquid in tank to height h is

$$W = \int_a^b \rho g A(y)(h - y)\,dy$$

Example (Energy to fill water tower)

Water tower has conical tank with bottom point at height of 20 m and top at 30 m with a top radius of 5 m. Give and integral for amount of work to fill the tank with water. (Density of water is 1000 $\mathrm{kg/m^3}$)

- At distance d from bottom of tank the cross-section of the tank is a circle with radius r.
- By similar triangles $\frac{d}{10} = \frac{r}{5}$ so $r = \frac{d}{2}$
- $d + 20 = y$ so $d = y - 20$.
- Hence cross-sectional area of tank at height y is

$$A(y) = \pi r^2 = \pi(\tfrac{d}{2})^2 = \pi \cdot (\tfrac{y-20}{2})^2$$

$$\text{Work to fill tank (in joules)} = \int_{20}^{30} \rho g A(y) \cdot (y - 0)\,dy$$
$$= \int_{20}^{30} 1000(9.8)\pi \cdot (\tfrac{y-20}{2})^2 \cdot y\,dy$$

Force on a dam

Force on a dam

1. width of dam at height y is $w(y)$.
2. height of bottom of dam is $y = a$
3. height of top of liquid is $y = b$
4. density of liquid is ρ (mass)/(unit volume).
5. acceleration of gravity is g

Then the **hydrostatic pressure** at height y is $\rho g(b - y)$ and the **total force on the dam** is

$$F = \int_a^b \rho g(b - y) w(y) \, dy$$

Example (Force on a dam)

Trapezoidal dam is $10\,\mathrm{m}$ wide at top and $1\,\mathrm{m}$ wide at bottom. The reservior that it supports is full to the top of the dam and is $9\,\mathrm{m}$ deep. Give an integral for total force on the dam. (Density of water is $1000\,\mathrm{kg}/\mathrm{m}^3$)

- Let bottom of dam be at height $y = 0$ to top is at $y = 9$.
- Width of trapezoid is linear in height so $w(y)$ is straight line function $w(y) = my + b$ with $w(0) = 1$ and $w(9) = 10$.
- Thus $w(y) = \frac{10-1}{9-0}(y - 0) + 1 = y + 1$.
- Total force on the dam is then

$$\begin{aligned}\text{Force on dam} &= \int_0^9 \rho g(9 - y) w(y) \, dy \\ &= \int_0^9 1000(9.8)(9 - y)(y + 1) \, dy\end{aligned}$$

Variable gravity

Newton's universal law of gravitation

The force of gravity between an object with mass m_1 and an object with mass m_2 whose centers of mass are distance r from eachother is

$$F = \frac{Gm_1 m_2}{r^2}$$

where $G = 6.674 \cdot 10^{-11}\,\mathrm{Nm}^2/\mathrm{kg}^2$

Example (Lifting far from Earth)

Give an integral for total work required to lift a $m = 10\,\mathrm{kg}$ weight to a height of $h = 600\,\mathrm{km} = 6 \cdot 10^5\mathrm{m}$.

- Mass of Earth $M = 5.9722 \cdot 10^{24}\,\mathrm{kg}$
- Earth radius $R = 6.3781 \cdot 10^6\,\mathrm{m}$

$$\begin{aligned}\text{Work to lift mass} &= \int_R^{R+h} \frac{GMm}{r^2}\,\mathrm{d}r \\ &= \int_{6.3781\cdot 10^6}^{6.3781\cdot 10^6 + 6\cdot 10^5} \frac{6.674 \cdot 10^{-11} \cdot 5.9722 \cdot 10^{24} \cdot 10}{r^2}\,\mathrm{d}r\end{aligned}$$