Lecture 34 – November 16, 2018

Announcements

- Office hours Mon. & Wed. 12:30pm-2pm in Math Tower (MW) room 650

Today

- Redefinition of natural logarithm and exponential functions

1. Previously we defined e in Section 3.3 as unique number satisfying $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.
2. But what does it mean to raise e to arbitrary real number? For example, how do we define $e^{\sqrt{2}}$?
3. Integrals area or symmetry
4. Today we'll give usable definition for natrual logarithm and then use it to define exponential function too.

(Re)definition of natural logarithm

Definition (Natural logarithm)

For $x \in (0, \infty)$

$$\ln x = \int_1^x \frac{1}{t} \, \mathrm{d}t$$

Theorem (Properties of the natural logarithm)

1. $\ln 1 = 0$.
2. *If* $x \neq 0$ *then* $\frac{\mathrm{d}}{\mathrm{d}x} \ln |x| = \frac{1}{x}$.
3. $\ln x$ *is differentiable and hence continuous on* $(0, \infty)$.
4. $\ln x$ *is increasing and concave down on* $(0, \infty)$.
5. $\ln(xy) = \ln x + \ln y$ *if* $x, y > 0$.
6. $\ln(\frac{x}{y}) = \ln x - \ln y$ *if* $x, y > 0$.
7. $\lim_{x \to \infty} \ln x = \infty$.
8. $\lim_{x \to 0^+} \ln x = -\infty$.
9. $\ln(x^p) = p \ln x$ *if* $x > 0$ *and* p *is any real number.*

Proof of 1.

- By definition $\ln 1 = \int_1^1 \frac{1}{t} \, \mathrm{d}t = 0$.

□

Proof of 2 and 3.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \ln |x| &= \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} \ln x, & x > 0 \\ \frac{\mathrm{d}}{\mathrm{d}x} \ln(-x), & x < 0 \end{cases} \\
&= \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} \int_1^x \frac{1}{t} \, \mathrm{d}t, & x > 0 \\ \frac{\mathrm{d}}{\mathrm{d}x} \int_1^{-x} \frac{1}{t} \, \mathrm{d}t, & x < 0 \end{cases} \\
&= \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{-x} \cdot (-1), & x < 0 \end{cases} \\
&= \frac{1}{x}
\end{aligned}$$

□

Proof of 4.

- By 2 we have $\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$.
- If $x > 0$ then $\frac{1}{x} > 0$ thus $\ln x$ is increasing on $(0, \infty)$.
- $\frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln x = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$.
- If $x > 0$ then $-\frac{1}{x^2} < 0$ thus $\ln x$ is concave down on $(0, \infty)$.

□

Proof of 5.

Suppose $x, y > 0$.

$$\begin{aligned}
\ln(xy) &= \int_1^{xy} \frac{1}{t}\,\mathrm{d}t \\
&= \int_1^{x} \frac{1}{t}\,\mathrm{d}t + \int_x^{xy} \frac{1}{t}\,\mathrm{d}t \\
u = \tfrac{t}{x}, &\qquad \mathrm{d}u = \tfrac{1}{x}\,\mathrm{d}t \\
&= \int_1^{x} \frac{1}{t}\,\mathrm{d}t + \int_1^{y} \frac{1}{u}\,\mathrm{d}u \\
&= \ln x + \ln y
\end{aligned}$$

□

Proof of 6.

Suppose $x, y > 0$.

$$\begin{aligned} \ln x &= \ln(\tfrac{x}{y} \cdot y) \\ &= \ln(\tfrac{x}{y}) + \ln y \end{aligned}$$

So solving for $\ln(\frac{x}{y})$ we get

$$\ln(\tfrac{x}{y}) = \ln x - \ln y$$

□

Proof of 7 and 8.

- For all $x \in [1, 2]$ we have $\frac{1}{x} \geq \frac{1}{2}$
- Thus $\ln 2 = \int_1^2 \frac{1}{x}\, dx \geq \int_1^2 \frac{1}{2}\, dx = \frac{1}{2}$
- CLAIM: For all natural numbers $n > 0$ we have $\ln(2^n) > \frac{n}{2}$.
- BASE CASE: For $n = 1$ we saw above that $\ln 2 \geq \frac{1}{2}$.
- INDUCTIVE STEP: Suppose $\ln(2^n) > \frac{n}{2}$.
- Then

$$\ln(2^{n+1}) = \ln(2^n \cdot 2) = \ln(2^n) + \ln 2 \geq \tfrac{n}{2} + \tfrac{1}{2} = \tfrac{n+1}{2}$$

- Now given $M > 0$ choose $N > 2^{2M}$.
- If $x > N$ then $\ln x > \ln N > \ln(2^{2M}) > \frac{2M}{2} = M$.
- Thus $\lim_{x \to \infty} \ln x = \infty$.
- Similarly given $M > 0$ again choose $0 < \delta < 2^{-2M}$.
- If $x - 0 > \delta$ then $\ln x < \ln \delta < \ln(2^{-2M}) < -\frac{2M}{2} = -M$.
- Thus $\lim_{x \to 0^+} \ln x = -\infty$.

□

(Re)definition of exponetial function

Definition (Exponential function)

$\exp x$ is the inverse function of $\ln x$

Theorem (Properties of the exponential function)

1. $\exp 0 = 1$.
2. *Domain of* $\exp x$ *is* $(-\infty, \infty)$ *and image is* $(0, \infty)$
3. $\frac{\mathrm{d}}{\mathrm{d}x} \exp x = \exp x$.
4. $\exp x$ *is differentiable and hence continuous on* $(-\infty, \infty)$.
5. $\exp x$ *is increasing and concave up on* $(-\infty, \infty)$.
6. $\exp(x + y) = \exp(x) \exp(y)$
7. $\exp(x - y) = \frac{\exp x}{\exp y}$
8. $\ln(\exp x) = x$
9. $\exp \ln x = x$ *if* $x > 0$.

Theorem (e exists and is unique)

There is a unique number $e \in (1, 4]$ *such that* $\ln e = 1$.

Proof.

- $\ln 1 = 0$
- $\ln 4 = \ln 2^2 \geq \frac{2}{2} = 1$.
- If $\ln 4 = 1$ then $e = 4$.
- Otherwise $\ln 4 > 1$ so by IVT there is $k \in (1, 4)$ such that $\ln k = 1$.
- $\ln x$ is increasing so there can be only one number k such that $\ln k = 1$.

□

Definition (General exponentiation)

If $a > 0$ then

$$a^b = \exp(b \ln a)$$

Theorem (exp is the exponential function with base e)

For all x we have $e^x = \exp x$

Proof.

$$e^x = \exp(x \ln e) = \exp(x \cdot 1) = \exp x$$

□

- Growth rate is y'
- Relative growth rate is $\frac{y'}{y}$
- Differential equation $y' = ky$. Solution is a function.
- $y = y_0 e^{kt}$ is solution with initial value y_0
- Doubling time $T_2 = \frac{\ln 2}{k}$
- APY of 5% means $y(1) = y_0 e^k = y_0 1.05$ so $k = \ln 1.05$