Lecture 35 – November 19, 2018

Announcements

- No office hours today

Today

- Integration by parts

Integration by parts

Product rule for differentiation

$$\frac{d}{dx}\left(u(x)v(x)\right) = u'(x)v(x) + u(x)v'(x)$$

- So

$$\begin{aligned} u(x)v(x) + C &= \int u'(x)v(x) + u(x)v'(x)\,dx \\ &= \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx \end{aligned}$$

- Solving for $\int u(x)v'(x)\,dx$ we get the equation for integration by parts

Integration by parts

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx$$

or

$$\int u\,dv = uv - \int v\,du$$

Example

Simplify $\int xe^{5x}\,dx$

- $u = x,\ dv = e^{5x}\,dx,\ du = dx,\ v = \frac{1}{5}e^{5x}$
- $$\int xe^{5x}\,dx = \tfrac{1}{5}xe^{5x} - \int \tfrac{1}{5}e^{5x}\,dx = \tfrac{1}{5}xe^{5x} - \tfrac{1}{25}e^{5x} + C$$
- CHECK:

$$\tfrac{d}{dx}\left(\tfrac{1}{5}xe^{5x} - \tfrac{1}{25}e^{5x} + C\right) = \tfrac{1}{5}e^{5x} + xe^{5x} - \tfrac{1}{5}e^{5x} = xe^{5x}\checkmark$$

Example

Simplify $\int x^3e^{2x}\,dx$

- $u = x^3,\ dv = e^{2x}\,dx,\ du = 3x^2\,dx,\ v = \frac{1}{2}e^{2x}$
- $$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - \int \tfrac{3}{2}x^2e^{2x}\,dx$$
- $u = 3x^2,\ dv = \frac{1}{2}e^{2x}\,dx,\ du = 6x\,dx,\ v = \frac{1}{4}e^{2x}$

$$\int x^3e^{2x}\,dx = \tfrac{1}{2}x^3e^{2x} - (\tfrac{3}{4}x^2e^{2x} - \int \tfrac{6}{4}xe^{2x}\,dx)$$

- $u = 6x,\ dv = \frac{1}{4}e^{2x}\,dx,\ du = 6\,dx,\ v = \frac{1}{8}e^{2x}$

$$\begin{aligned}\int x^3e^{2x}\,dx &= \tfrac{1}{2}x^3e^{2x} - (\tfrac{3}{4}x^2e^{2x} - (\tfrac{6}{8}xe^{2x} - \int \tfrac{6}{8}e^{2x}\,dx)) \\ &= \tfrac{1}{2}x^3e^{2x} - \tfrac{3}{4}x^2e^{2x} + \tfrac{6}{8}xe^{2x} - \tfrac{6}{16}e^{2x} + C\end{aligned}$$

Tabular integration

The problem above illustrates how some problems require multiple integrations by parts. Tabular integration in a quick way to do this.

Example

Calculate $\int x^3 e^{2x}\,\mathrm{d}x$ using tabular integration

D		I
x^3	$+$	e^{2x}
$3x^2$	$-$	$\frac{1}{2}e^{2x}$
$6x$	$+$	$\frac{1}{4}e^{2x}$
6	$-$	$\frac{1}{8}e^{2x}$
0	$+$	$\frac{1}{16}e^{2x}$

$$\begin{aligned}\int x^3 e^{2x}\,\mathrm{d}x &= x^3\cdot\tfrac{1}{2}e^{2x} - 3x^2\cdot\tfrac{1}{4}e^{2x} + 6x\cdot\tfrac{1}{8}e^{2x} - 6\cdot\tfrac{1}{16}e^{2x} + \int 0\cdot\tfrac{1}{16}e^{2x}\,\mathrm{d}x\\ &= x^3\cdot\tfrac{1}{2}e^{2x} - 3x^2\cdot\tfrac{1}{4}e^{2x} + 6x\cdot\tfrac{1}{8}e^{2x} - 6\cdot\tfrac{1}{16}e^{2x} + C\end{aligned}$$

Example

Calculate $\int x^2 \cos 3x \, dx$ using tabular integration

D		I
x^2	$+$	$\cos 3x$
$2x$	$-$	$\frac{1}{3}\sin 3x$
2	$+$	$-\frac{1}{9}\cos 3x$
0	$-$	$-\frac{1}{27}\sin 3x$

$$\int x^3 e^{2x} \, dx = x^2 \cdot \tfrac{1}{3}\sin 3x - 2x \cdot \left(-\tfrac{1}{9}\cos 3x\right) + 2 \cdot \left(-\tfrac{1}{27}\sin 3x\right) - \int 0 \cdot \left(-\tfrac{1}{27}\sin 3x\right) dx$$

$$= x^2 \cdot \tfrac{1}{3}\sin 3x - 2x \cdot \left(-\tfrac{1}{9}\cos 3x\right) + 2 \cdot \left(-\tfrac{1}{27}\sin 3x\right) + C$$

Example

Calculate $\int e^{4x} \sin 3x \, dx$ using tabular integration

D		I
$\sin 3x$	$+$	e^{4x}
$3\cos 3x$	$-$	$\frac{1}{4}e^{4x}$
$-9\sin 3x$	$+$	$\frac{1}{16}e^{4x}$

$$\int e^{4x}\sin 3x \, dx = \sin 3x \cdot \tfrac{1}{4}e^{4x} - 3\cos 3x \cdot \tfrac{1}{16}e^{4x} + \int (-9\sin 3x) \cdot \tfrac{1}{16}e^{4x} \, dx$$

$$\int e^{4x}\sin 3x \, dx = \sin 3x \cdot \tfrac{1}{4}e^{4x} - 3\cos 3x \cdot \tfrac{1}{16}e^{4x} - \tfrac{9}{16}\int \sin 3x \cdot e^{4x} \, dx$$

$$\int e^{4x}\sin 3x \, dx + \tfrac{9}{16}\int e^{4x}\sin 3x \, dx = \sin 3x \cdot \tfrac{1}{4}e^{4x} - 3\cos 3x \cdot \tfrac{1}{16}e^{4x}$$

$$\tfrac{25}{16}\int e^{4x}\sin 3x \, dx = \sin 3x \cdot \tfrac{1}{4}e^{4x} - 3\cos 3x \cdot \tfrac{1}{16}e^{4x} + C_1$$

$$\int e^{4x}\sin 3x \, dx = \tfrac{16}{25}\left(\sin 3x \cdot \tfrac{1}{4}e^{4x} - 3\cos 3x \cdot \tfrac{1}{16}e^{4x}\right) + C$$

A few more integration by parts problems

Example

Simplify $\int \ln x \, dx$

- $u = \ln x,\ dv = dx,\ du = \frac{1}{x}\,dx,\ v = x$
- $$\begin{aligned}\int \ln x\,dx &= x\ln x - \int \tfrac{1}{x}\cdot x\,dx\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + C\end{aligned}$$

Example

Simplify $\int \arctan x \, dx$

- $u = \arctan x,\ dv = dx,\ du = \frac{1}{1+x^2}\,dx,\ v = x$
- $$\begin{aligned}\int \arctan x\,dx &= x\arctan x - \int \tfrac{1}{1+x^2}\cdot x\,dx\\ & w = 1+x^2, \quad dw = 2x\,dx\\ &= x\arctan x - \int \tfrac{1}{w}\cdot\tfrac{1}{2}\,dw\\ &= x\arctan x - \tfrac{1}{2}\ln|w| + C\\ &= x\arctan x - \tfrac{1}{2}\ln|1+x^2| + C\end{aligned}$$

Example

Simplify $\int (\ln x)^3 \, dx$

- $u = (\ln x)^3, \; dv = dx, \; du = \frac{3(\ln x)^2}{x} \, dx, \; v = x$
- $$\int (\ln x)^3 \, dx = x(\ln x)^3 - \int \frac{3(\ln x)^2}{x} \cdot x \, dx$$ $$= x(\ln x)^3 - \int 3(\ln x)^2 \, dx$$

- $u = 3(\ln x)^2, \; dv = dx, \; du = \frac{6(\ln x)}{x} \, dx, \; v = x$
- $$\int (\ln x)^3 \, dx = x(\ln x)^3 - \left(x \cdot 3(\ln x)^2 - \int \frac{6(\ln x)}{x} \cdot x \, dx \right)$$ $$= x(\ln x)^3 - 3x(\ln x)^2 + \int 6 \ln x \, dx$$ $$= x(\ln x)^3 - 3x(\ln x)^2 + 6x \ln x - 6x + C$$