Lecture 37 – November 28, 2018

Announcements

- Office hours Mon. & Wed. 12:30pm-2pm in Math Tower (MW) room 650
- Final Exam 8pm-9:45pm on Monday 12/10/2018 in Hitchcock Hall (HI) room 031
- Email me if you have a conflict
- Written Homework 7 posted
- Final studyguide posted

Today

- Trigonometric substitutions

Trigonometric substitutions

Integral involves	Substitution	Domain of validity
$a^2 - x^2$	$x = a \sin\theta$	$\lvert x \rvert \leq a$
$a^2 + x^2$	$x = a \tan\theta$	$x \in \mathbf{R}$
$x^2 - a^2$	$x = a \sec\theta$	$x \geq a$ or $x \leq -a$ (but not both)

Note that these are "reverse" substitutions in that $x = f(\theta)$ instead of the usual $u = g(x)$.

Example

Let's start with an integration that we can perform with or without a trigonometric substitution.

$\int \frac{dx}{4+x^2}$

1. First we will integrate using a trig subsitution

$$x = 2\tan\theta, \quad dx = 2\sec^2\theta\, d\theta$$

$$\begin{aligned}
\int \frac{dx}{4+x^2} &= \int \frac{2\sec^2\theta\, d\theta}{4+(2\tan\theta)^2} \\
&= \int \frac{2\sec^2\theta\, d\theta}{4(1+\tan^2\theta)} \\
&= \tfrac{1}{2}\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} \\
&= \tfrac{1}{2}\int d\theta \\
&= \tfrac{\theta}{2} + C
\end{aligned}$$

$$x = 2\tan\theta \text{ so } \tfrac{x}{2} = \tan\theta \text{ so } \arctan\tfrac{x}{2} = \theta$$

$$= \tfrac{1}{2}\arctan\tfrac{x}{2} + C$$

Example (*continued*)

2. Next we will integrate without a trig subsitution

$$\int \frac{dx}{4+x^2} = \tfrac{1}{4}\int \frac{dx}{1+(\frac{x}{2})^2}$$

$$u = \tfrac{x}{2}, \quad du = \tfrac{1}{2}\,dx$$

$$\begin{aligned}
&= \tfrac{1}{4}\int \frac{2\,du}{1+u^2} \\
&= \tfrac{1}{2}\arctan u + C \\
&= \tfrac{1}{2}\arctan\tfrac{x}{2} + C
\end{aligned}$$

Recall the reduction formulas for trigonometric integrals from Monday's lecture:

Reduction formulas *(will be provided on exams)*

If n is a positive integer then

$$\int \sin^n x\,dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx, \qquad n \neq 1$$

$$\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx, \qquad n \neq 1$$

$$\int \cot^n x\,dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x\,dx, \qquad n \neq 1$$

$$\int \csc^n x\,dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\,dx, \qquad n \neq 1$$

Example

$\int \sqrt{1-x^2}\,dx$

$$x = \sin\theta, \quad dx = \cos\theta\,d\theta$$

$$\begin{aligned}
\int \sqrt{1-x^2}\,dx &= \int \sqrt{1-(\sin\theta)^2}\cos\theta\,d\theta \\
&= \int \sqrt{\cos^2\theta}\cos\theta\,d\theta \\
&= \int \cos^2\theta\,d\theta \\
&\text{using cos reduction formula} \\
&= \frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int 1\,d\theta \\
&= \frac{\cos\theta\sin\theta}{2} + \frac{\theta}{2} + C \\
&x = \sin\theta \text{ so } \theta = \arcsin x \text{ and } \cos\theta = \sqrt{1-x^2} \\
&= \frac{(\sqrt{1-x^2})x}{2} + \frac{\arcsin x}{2} + C
\end{aligned}$$

Note that the domain of validity for this antiderivative calculation is $x \in [-1, 1]$

Example

$\int \frac{dx}{\sqrt{x^2+9}}$

$$x = 3\tan\theta, \quad dx = 3\sec^2\theta\, d\theta$$

$$\begin{aligned}
\int \frac{dx}{\sqrt{x^2+9}} &= \int \frac{3\sec^2\theta\, d\theta}{\sqrt{(3\tan\theta)^2+9}} \\
&= \int \frac{3\sec^2\theta\, d\theta}{\sqrt{9((\tan\theta)^2+1)}} \\
&= \int \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}} \\
&= \int \sec\theta\, d\theta \\
&= \ln|\sec\theta + \tan\theta| + C \\
x = 3\tan\theta &\text{ so } \tfrac{x}{3} = \tan\theta \text{ so } \sec\theta = \tfrac{\sqrt{9+x^2}}{3} \\
&= \ln\left|\tfrac{\sqrt{9+x^2}}{3} + \tfrac{x}{3}\right| + C
\end{aligned}$$

Completing the square

If your quadratic (function of the form $ax^2 + bx + c$) has nonzero bx term then you will need to **complete the square**

$$\begin{aligned}
ax^2 + bx + c &= a\left(x^2 + \tfrac{b}{a}\cdot x + \tfrac{c}{a}\right) \\
&= a\left(x^2 + \tfrac{b}{a}\cdot x + \tfrac{b^2}{4a^2} - \tfrac{b^2}{4a^2} + \tfrac{c}{a}\right) \\
&= a\left(\left(x + \tfrac{b}{2a}\right)^2 - \tfrac{b^2}{4a^2} + \tfrac{c}{a}\right)
\end{aligned}$$

Example

Complete the square for $3x^2 - 6x + 10$.

$$\begin{aligned}
3x^2 - 6x + 10 &= 3(x^2 - 2x + \tfrac{10}{3}) \\
&= 3(x^2 - 2x + 1 - 1 + \tfrac{10}{3}) \\
&= 3((x-1)^2 - 1 + \tfrac{10}{3}) \\
&= 3((x-1)^2 + \tfrac{7}{3})
\end{aligned}$$

Example

$\int_4^5 \sqrt{x^2 - 6x + 8}\,\mathrm{d}x$

$$\begin{aligned} x^2 - 6x + 8 &= x^2 - 6x + 9 - 9 + 8 \\ &= (x-3)^2 - 1 \end{aligned}$$

$$\begin{aligned} \int_4^5 \sqrt{x^2 - 6x + 8}\,\mathrm{d}x &= \int_4^5 \sqrt{(x-3)^2 - 1}\,\mathrm{d}x \\ & u = x - 3, \quad \mathrm{d}u = \mathrm{d}x \\ &= \int_{4-3}^{5-3} \sqrt{u^2 - 1}\,\mathrm{d}u \\ & u = \sec\theta, \quad \mathrm{d}u = \sec\theta\tan\theta\,\mathrm{d}\theta \\ &= \int_{u=1}^{2} \sqrt{(\sec\theta)^2 - 1}\sec\theta\tan\theta\,\mathrm{d}\theta \\ &= \int_{u=1}^{2} \sqrt{\tan^2\theta}\sec\theta\tan\theta\,\mathrm{d}\theta \\ &= \int_{u=1}^{2} \sec\theta\tan^2\theta\,\mathrm{d}\theta \end{aligned}$$

Example (*continued*)

$$= \int_{u=1}^{2} \sec\theta(\sec^2\theta - 1)\theta\,\mathrm{d}\theta$$

$$= \int_{u=1}^{2} \sec^3\mathrm{d}\theta - \int_{u=1}^{2} \sec\theta\,\mathrm{d}\theta$$

secant reduction

$$= \frac{\sec\theta\tan\theta}{2}\Big|_{u=1}^{2} + \frac{1}{2}\int_{u=1}^{2} \sec\theta\,\mathrm{d}\theta - \int_{u=1}^{2} \sec\theta\,\mathrm{d}\theta$$

$$= \frac{\sec\theta\tan\theta}{2}\Big|_{u=1}^{2} - \frac{1}{2}\int_{u=1}^{2} \sec\theta\,\mathrm{d}\theta$$

$$= \frac{\sec\theta\tan\theta}{2}\Big|_{u=1}^{2} - \frac{1}{2}\ln|\sec\theta + \tan\theta|\Big|_{u=1}^{2}$$

$u = \sec\theta$ so $\tan\theta = \sqrt{u^2 - 1}$

$$= \frac{u\sqrt{u^2-1}}{2}\Big|_{u=1}^{2} - \frac{1}{2}\ln\left|u + \sqrt{u^2-1}\right|\Big|_{u=1}^{2}$$

$$= \left(\frac{2\sqrt{2^2-1}}{2} - \frac{1\sqrt{1^2-1}}{2}\right) - \left(\frac{1}{2}\ln\left|2 + \sqrt{2^2-1}\right| - \frac{1}{2}\ln\left|1 + \sqrt{1^2-1}\right|\right)$$

Example

$\int \frac{\mathrm{d}x}{(4-x^2)^3}$

$$\int \frac{\mathrm{d}x}{(4-x^2)^3} = \int \frac{\mathrm{d}x}{(2^2-x^2)^3}$$

$$x = 2\sin\theta, \quad \mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$$

$$= \int \frac{2\cos\theta\,\mathrm{d}\theta}{(2^2-(2\sin\theta)^2)^3}$$

$$= \int \frac{2\cos\theta\,\mathrm{d}\theta}{(4-4\sin^2\theta)^3}$$

$$= \int \frac{2\cos\theta\,\mathrm{d}\theta}{4^3(1-\sin^2\theta)^3}$$

$$= \int \frac{2\cos\theta\,\mathrm{d}\theta}{4^3(\cos^2\theta)^3}$$

$$= \frac{1}{32}\int \sec^5\theta\,\mathrm{d}\theta$$

secant reduction

$$= \frac{1}{32}\left(\frac{\sec^3\theta\tan\theta}{4} + \frac{3}{4}\int \sec^3\theta\,\mathrm{d}\theta\right)$$

Example (*continued*)

secant reduction

$$= \frac{1}{32}\left(\frac{\sec^3\theta\tan\theta}{4} + \frac{3}{4}\left(\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int \sec\theta\,\mathrm{d}\theta\right)\right)$$

$$= \frac{1}{32}\left(\frac{\sec^3\theta\tan\theta}{4} + \frac{3}{4}\left(\frac{\sec\theta\tan\theta}{2} + \frac{\ln|\sec\theta+\tan\theta|}{2}\right)\right) + C$$

$x = 2\sin\theta$ so $\frac{x}{2} = \sin\theta$ so $\sec\theta = \frac{2}{\sqrt{4-x^2}}$ and $\tan\theta = \frac{x}{\sqrt{4-x^2}}$

$$= \frac{1}{32\cdot 4}\left(\frac{2}{\sqrt{4-x^2}}\right)^3\left(\frac{x}{\sqrt{4-x^2}}\right) + \frac{3}{32\cdot 4\cdot 2}\left(\frac{2}{\sqrt{4-x^2}}\right)\left(\frac{x}{\sqrt{4-x^2}}\right) + \frac{3}{32\cdot 4\cdot 2}\ln\left|\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}}\right| + C$$