# Lecture 38 – November 30, 2018

## Announcements

- Office hours Mon. & Wed. 12:30pm-2pm in Math Tower (MW) room 650
- Final Exam 8pm-9:45pm on Monday 12/10/2018 in Hitchcock Hall (HI) room 031
- Email me if you have a conflict
- Written Homework 7 posted
- Final studyguide posted

## Today

- Partial fractions

**Partial fractions** is a format for expressing a rational function $\frac{p(x)}{q(x)}$ that is helps with integration.

## Partial fractions

1. If the degree of the numerator is greater than or equal to the degree of the denominator perform **long division** in order to express your rational function
$$\frac{p(x)}{q(x)} = s(x) + \frac{r(x)}{q(x)}$$
where $s(x)$ and $r(x)$ are polynomials with degree of $r(x)$ strictly less than that of $q(x)$.
2. Factor the denominator into **linear** factors (factors of the form $x - a$) and **irreducible quadratic** factors (factors of the form $x^2 + bx + c$ with $b^2 - 4c < 0$)

$$q(x) = (x - a_1)^{m_1}(x - a_2)^{m_2} \cdots (x - a_k)^{m_k}(x^2 + b_1x + c_1)^{n_1} \cdot (x^2 + b_2x + c_2)^{n_2} \cdots (x^2 + b_\ell x + c_\ell)^{n_\ell}$$

## Partial fractions (*continued*)

3. Express $\frac{r(x)}{q(x)}$ as a sum of the form

$$\begin{aligned}\frac{r(x)}{q(x)} &= \frac{A_{1,1}}{x-a_1} + \frac{A_{1,2}}{(x-a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x-a_1)^{m_1}}\\ &+ \frac{A_{2,1}}{x-a_2} + \frac{A_{2,2}}{(x-a_2)^2} + \cdots + \frac{A_{2,m_2}}{(x-a_2)^{m_2}}\\ &\vdots\\ &+ \frac{A_{k,1}}{x-a_k} + \frac{A_{k,2}}{(x-a_k)^2} + \cdots + \frac{A_{k,m_k}}{(x-a_k)^{m_k}}\\ &+ \frac{B_{1,1}x+C_{1,1}}{x^2+b_1x+c_1} + \frac{B_{1,2}x+C_{1,2}}{(x^2+b_1x+c_1)^2} + \cdots + \frac{B_{1,n_1}x+C_{1,n_1}}{(x^2+b_1x+c_1)^{n_1}}\\ &+ \frac{B_{2,1}x+C_{2,1}}{x^2+b_2x+c_2} + \frac{B_{2,2}x+C_{2,2}}{(x^2+b_2x+c_2)^2} + \cdots + \frac{B_{2,n_2}x+C_{2,n_2}}{(x^2+b_2x+c_2)^{n_2}}\\ &\vdots\\ &+ \frac{B_{\ell,1}x+C_{\ell,1}}{x^2+b_\ell x+c_\ell} + \frac{B_{\ell,2}x+C_{\ell,2}}{(x^2+b_\ell x+c_\ell)^2} + \cdots + \frac{B_{\ell,n_\ell}x+C_{\ell,n_\ell}}{(x^2+b_\ell x+c_\ell)^{n_\ell}}\end{aligned}$$

4. Solve for unknowns $A_{1,1}, \cdots, C_{\ell,n_\ell}$.

## Example

Express the rational function $\frac{x^3+6x^2-9x-86}{x^2+2x-15}$ using partial fractions.

- polynomial division gives

$$\frac{x^3+6x^2-9x-86}{x^2+2x-15} = x+4+\frac{-2x-26}{x^2+2x-15}$$

- Denominator factors as $x^2+2x-15=(x+5)(x-3)$
-

$$\begin{aligned}\frac{-2x-26}{x^2+2x-15} &= \frac{A}{x+5}+\frac{B}{x-3}\\ -2x-26 &= A(x-3)+B(x+5)\\ -2x-26 &= Ax-3A+Bx+5B\end{aligned}$$

- Coefficient of $x$ gives equation: $-2 = A+B$
  Constant terms give equation: $-26 = -3A+5B$
- Solving for $A$ and $B$ we get $A=2$ and $B=-4$
- Thus $\frac{-2x-26}{x^2+2x-15} = \frac{2}{x+5}+\frac{-4}{x-3}$
- Thus $\frac{x^3+6x^2-9x-86}{x^2+2x-15} = x+4+\frac{2}{x+5}+\frac{-4}{x-3}$

## Example

Now integrate $\int \frac{x^3+6x^2-9x-86}{x^2+2x-15}\,\mathrm{d}x$

$$\begin{aligned}\int \frac{x^3+6x^2-9x-86}{x^2+2x-15}\,\mathrm{d}x &= \int x+4+\frac{2}{x+5}+\frac{-4}{x-3}\,\mathrm{d}x \\ &= \frac{x^2}{2}+4x+2\ln|x+5|-4\ln|x-3|+C\end{aligned}$$

## Example

Express the rational function $\frac{x^2-2x-3}{(x-2)^3}$ using partial fractions.

- $$\begin{aligned}\frac{x^2-2x-3}{(x-2)^3} &= \frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{(x-2)^3} \\ x^2-2x-3 &= A(x-2)^2+B(x-2)+C \\ x^2-2x-3 &= Ax^2-4Ax+4A+Bx-2B+C\end{aligned}$$
- Coefficient of $x^2$ gives equation: $1 = A$
  Coefficient of $x$ gives equation: $-2 = -4A + B$
  Constant terms give equation: $-3 = 4A - 2B + C$
- Solving for $A$ and $B$ we get $A = 1$, $B = 2$ and $C = -3$
- Thus $\frac{x^2-2x-3}{(x-2)^3} = \frac{1}{x-2}+\frac{2}{(x-2)^2}+\frac{-3}{(x-2)^3}$

## Example

Now integrate $\int \frac{x^2-2x-3}{(x-2)^3}\,\mathrm{d}x$

$$\begin{aligned}\int \frac{x^2-2x-3}{(x-2)^3}\,\mathrm{d}x &= \int \frac{1}{x-2} + \frac{2}{(x-2)^2} + \frac{-3}{(x-2)^3}\,\mathrm{d}x \\ &= \ln|x-2| - \frac{1}{x-2} + \frac{3}{2(x-2)^2} + C\end{aligned}$$

## Example

What format should the partial fractions expression for

$$\frac{5x^2-7x-10}{x(x-3)^2(x^2+4)(x^2+6x+18)^3}$$

take? Don't solve for the coefficients.

$$\begin{aligned}&\frac{5^2-7x-10}{(x+1)(x-3)^2(x^2+4)(x^2+6x+18)^2} \\ &= \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{x^2+6x+18} + \frac{Hx+J}{(x^2+6x+18)^2} + \frac{Kx+L}{(x^2+6x+18)^3}\end{aligned}$$

## Example

These are the 4 types of integrals which might arise when integrating a function written in partial fractions format:

1. $\int \frac{2}{x-4}\,dx$

$$\int \frac{2}{x-4}\,dx = 2\ln|x-4| + C$$

2. $\int \frac{2}{(x-4)^4}\,dx$

$$\int \frac{2}{(x-4)^4}\,dx = \frac{2}{-3}(x-4)^{-3} + C$$

## Example (*continued*)

3. $\int \frac{x+5}{x^2+6x+18}\,dx$

$$\begin{aligned}
\int \frac{x+5}{x^2+6x+18}\,dx &= \int \frac{x+3}{x^2+6x+18}\,dx + \int \frac{-3+5}{x^2+6x+18}\,dx \\
&u = x^2+6x+18, \quad du = 2x+6\,dx \\
&= \int \frac{1}{u}\cdot\frac{1}{2}\,du + \int \frac{2}{(x+3)^2+9}\,dx \\
&w = x+3, \quad dw = dx \\
&= \frac{1}{2}\ln|u| + \int \frac{2}{w^2+9}\,dw \\
&= \frac{1}{2}\ln|x^2+6x+18| + \frac{2}{9}\int \frac{1}{\left(\frac{w}{3}\right)^2+1}\,dw \\
&y = \frac{w}{3}, \quad dy = \frac{1}{3}\,dw \\
&= \frac{1}{2}\ln|x^2+6x+18| + \frac{2}{9}\int \frac{1}{y^2+1}\cdot 3\,dy \\
&= \frac{1}{2}\ln|x^2+6x+18| + \frac{2\cdot 3}{9}\arctan y + C \\
&= \frac{1}{2}\ln|x^2+6x+18| + \frac{2}{3}\arctan\frac{w}{3} + C \\
&= \frac{1}{2}\ln|x^2+6x+18| + \frac{2}{3}\arctan\frac{x+3}{3} + C
\end{aligned}$$

## Example (*continued*)

4. $\int \frac{x+5}{(x^2+6x+18)^3}\,\mathrm{d}x$

$$\int \frac{x+5}{(x^2+6x+18)^3}\,\mathrm{d}x = \int \frac{x+3}{(x^2+6x+18)^3}\,\mathrm{d}x + \int \frac{-3+5}{(x^2+6x+18)^3}\,\mathrm{d}x$$

$u = x^2 + 6x + 18, \quad \mathrm{d}u = 2x + 6\,\mathrm{d}x$

$$= \int \frac{1}{u^3}\cdot\frac{1}{2}\,\mathrm{d}u + \int \frac{2}{\left((x+3)^2+9\right)^3}\,\mathrm{d}x$$

$x + 3 = 3\tan\theta, \quad \mathrm{d}x = 3\sec^2\theta\,\mathrm{d}\theta$

$$= \frac{u^{-2}}{2(-2)} + \int \frac{2\cdot 3\sec^2\theta\,\mathrm{d}\theta}{\left((3\tan\theta)^2+9\right)^3}$$

$$= \frac{1}{-4(x^2+6x+18)^2} + \frac{2\cdot 3}{9^3}\int \frac{\sec^2\theta\,\mathrm{d}\theta}{(\tan^2\theta+1)^3}$$

$$= \frac{1}{-4(x^2+6x+18)^2} + \frac{2}{3^5}\int \frac{\sec^2\theta\,\mathrm{d}\theta}{(\sec^2\theta)^3}$$

$$= \frac{1}{-4(x^2+6x+18)^2} + \frac{2}{3^5}\int \cos^4\theta\,\mathrm{d}\theta$$

cosine reduction

$$= \frac{1}{-4(x^2+6x+18)^2} + \frac{2}{3^5}\left(\frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\int \cos^2\theta\,\mathrm{d}\theta\right)$$

cosine reduction

$$= \frac{1}{-4(x^2+6x+18)^2} + \frac{2}{3^5}\left(\frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\left(\frac{\cos\theta\sin\theta}{2} + \frac{1}{2}\int \mathrm{d}\theta\right)\right)$$

$$= \frac{1}{-4(x^2+6x+18)^2} + \frac{2}{3^5}\left(\frac{\cos^3\theta\sin\theta}{4} + \frac{3}{4}\left(\frac{\cos\theta\sin\theta}{2} + \frac{\theta}{2}\right)\right) + C$$

$x + 3 = 3\tan\theta$ so $\frac{x+3}{3} = \tan\theta$ so $\sin\theta = \frac{x+3}{\sqrt{(x+3)^2+9}}$ and $\cos\theta = \frac{3}{\sqrt{(x+3)^2+9}}$

and $\theta = \arctan\frac{x+3}{3}$

$$= \frac{1}{-4(x^2+6x+18)^2} + \frac{2}{3^5\cdot 4}\left(\frac{3}{\sqrt{(x+3)^2+9}}\right)^3\left(\frac{x+3}{\sqrt{(x+3)^2+9}}\right)$$

$$+ \frac{2\cdot 3}{3^5\cdot 4\cdot 2}\left(\frac{3}{\sqrt{(x+3)^2+9}}\right)\left(\frac{x+3}{\sqrt{(x+3)^2+9}}\right) + \frac{2\cdot 3}{3^5\cdot 4\cdot 2}\left(\arctan\frac{x+3}{3}\right) + C$$