Lecture 39 – December 3, 2018

Announcements

- Office hours Mon. & Wed. 12:30pm-2pm in Math Tower (MW) room 650
- Final Exam 8pm-9:45pm on Monday 12/10/2018 in Hitchcock Hall (HI) room 031
- Email me if you have a conflict
- Written Homework 7 posted
- Final studyguide posted

Today

- Improper integrals

Definition (Value of an improper integral)

1. If f is continuous on $[a, \infty)$ then

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{B \to \infty} \int_a^B f(x)\,\mathrm{d}x$$

2. If f is continuous on $(-\infty, b]$ then

$$\int_{-\infty}^b f(x)\,\mathrm{d}x = \lim_{A \to -\infty} \int_A^b f(x)\,\mathrm{d}x$$

3. If f is continuous on $(-\infty, \infty)$ then

$$\int_{-\infty}^\infty f(x)\,\mathrm{d}x = \int_{-\infty}^c f(x)\,\mathrm{d}x + \int_c^\infty f(x)\,\mathrm{d}x$$

If any the relevant limits above do not exist (including if they are infinite) then the improper integral **diverges**.

Example

Evaluate the following improper integrals

- $\int_1^\infty \frac{1}{x^2}\,dx$

$$\begin{aligned}\int_1^\infty \tfrac{1}{x^2}\,dx &= \lim_{B\to\infty}\int_1^B \tfrac{1}{x^2}\,dx\\ &= \lim_{B\to\infty} -x^{-1}|_{x=1}^{B}\\ &= \lim_{B\to\infty} -B^{-1}+1^{-1}\\ &= 1\end{aligned}$$

- $\int_{-\infty}^{-2} \frac{1}{x}\,dx$

$$\begin{aligned}\int_1^\infty \tfrac{1}{x}\,dx &= \lim_{A\to-\infty}\int_A^{-2} \tfrac{1}{x}\,dx\\ &= \lim_{A\to-\infty} \ln|x||_{x=A}^{-2}\\ &= \lim_{A\to-\infty} \ln|-2|-\ln|A|\\ &\text{diverges}\end{aligned}$$

Example (*continued*)

- $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

$$\begin{aligned}\int_{-\infty}^\infty \tfrac{1}{x^2+1}\,dx &= \int_{-\infty}^0 \tfrac{1}{x^2+1}\,dx + \int_0^\infty \tfrac{1}{x^2+1}\,dx\\ &= \lim_{A\to-\infty}\int_A^0 \tfrac{1}{x^2+1}\,dx + \lim_{B\to\infty}\int_0^B \tfrac{1}{x^2+1}\,dx\\ &= \lim_{A\to-\infty} \arctan x\Big|_{x=A}^{0} + \lim_{B\to\infty}\arctan x\Big|_{x=0}^{B}\\ &= \lim_{A\to-\infty}(\arctan 0-\arctan A) + \lim_{B\to\infty}\arctan B-\arctan 0\\ &= 0-\left(-\tfrac{\pi}{2}\right)+\tfrac{\pi}{2}-0\\ &= \pi\end{aligned}$$

Example

How much work is required for an object of mass m to completely escape earth's gravity?

- The force of gravity between an object with mass M and an object with mass m whose centers of mass are distance r from each other is

$$F = \frac{GMm}{r^2}$$

- Universal gravitational constant $G = 6.674 \cdot 10^{-11}\,\mathrm{Nm^2/kg^2}$
- Mass of Earth $M = 5.9722 \cdot 10^{24}\,\mathrm{kg}$
- Earth radius $R = 6.3781 \cdot 10^{6}\,\mathrm{m}$

$$\begin{aligned}
\text{Work to escape Earth gravity} &= \int_R^\infty \frac{GMm}{r^2}\,\mathrm{d}r \\
&= \lim_{B\to\infty} \int_R^B \frac{GMm}{r^2}\,\mathrm{d}r \\
&= \lim_{B\to\infty} -GMmx^{-1}|_{x=R}^{B} \\
&= \lim_{B\to\infty} -GMmB^{-1} + GMmR^{-1} \\
&= \tfrac{GMm}{R}
\end{aligned}$$

Definition (Value of an improper integral)

1. If f is continuous on $[a, b)$ and $\lim_{x\to b^-} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{B\to b^-} \int_a^B f(x)\,\mathrm{d}x$$

2. If f is continuous on $(a, b]$ and $\lim_{x\to a^+} f(x) = \pm\infty$ then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{A\to a^+} \int_A^b f(x)\,\mathrm{d}x$$

3. If f is continuous on $[a, b]$ except at $c \in (a, b)$ then

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

If any the relevant limits above do not exist (including if they are infinite) then the improper integral **diverges**.

Example

Evaluate the following improper integrals

- $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x &= \lim_{A\to 0^+}\int_A^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x \\ &= \lim_{A\to 0^+} 2x^{\frac{1}{2}}\Big|_{x=A}^{1} \\ &= \lim_{A\to 0^+} 2\cdot 1^{\frac{1}{2}} - 2\cdot A^{\frac{1}{2}} \\ &= 2\end{aligned}$$

- $\int_{-2}^0 \frac{1}{x}\,\mathrm{d}x$

$$\begin{aligned}\int_{-2}^{\infty} \frac{1}{x}\,\mathrm{d}x &= \lim_{B\to 0^-}\int_{-2}^{B} \frac{1}{x}\,\mathrm{d}x \\ &= \lim_{B\to 0^-} \ln|x|\Big|_{x=-2}^{B} \\ &= \lim_{B\to 0^-} \ln|B| - \ln|-2| \\ &\text{diverges}\end{aligned}$$

Example (*continued*)

- $\int_{-1}^2 \frac{1}{x^3}\,\mathrm{d}x$

$$\begin{aligned}\int_{-1}^2 \frac{1}{x^3}\,\mathrm{d}x &= \int_{-1}^0 \frac{1}{x^3}\,\mathrm{d}x + \int_0^2 \frac{1}{x^3}\,\mathrm{d}x \\ &= \lim_{B\to 0^-}\int_{-1}^{B} \frac{1}{x^3}\,\mathrm{d}x + \lim_{A\to 0^+}\int_A^2 \frac{1}{x^3}\,\mathrm{d}x \\ &= \lim_{B\to 0^-} -\frac{x^{-2}}{2}\Big|_{-1}^{B} + \lim_{A\to 0^+} -\frac{x^{-2}}{2}\Big|_{x=A}^{2} \\ &= \lim_{B\to 0^-}\left(-\frac{B^{-2}}{2} + \frac{(-1)^{-2}}{2}\right) + \lim_{A\to 0^+}\left(-\frac{2^{-2}}{2} + \frac{A^{-2}}{2}\right) \\ &\text{diverges}\end{aligned}$$

Example

The equation for the upper half of the circle of radius r is

$$f(x) = \sqrt{r^2 - x^2}$$

Compute its arc length for $x \in [-r, r]$

- $f'(x) = \frac{1}{2}(r^2 - x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{r^2-x^2}}$
- Thus arc length is

$$L = \int_{-r}^{r} \sqrt{1 + (f'(x))^2}\,\mathrm{d}x = \int_{-r}^{r} \sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,\mathrm{d}x = \int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x$$

- First we find the antiderivative

$$\begin{aligned}
\int \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x =& \\
& x = r\sin\theta, \qquad \mathrm{d}x = r\cos\theta\,\mathrm{d}\theta \\
&= \int \sqrt{\frac{r^2}{r^2 - r^2\sin^2\theta}}\, r\cos\theta\,\mathrm{d}\theta \\
&= \int \sqrt{\frac{r^2}{r^2\cos^2\theta}}\, r\cos\theta\,\mathrm{d}\theta \\
&= \int r\,\mathrm{d}\theta \\
&= r\theta + C \\
&= r\arcsin\frac{x}{r} + C
\end{aligned}$$

- Notice that the definite integral $\int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x$ is an improper integral since the integrand has vertical asymptotes at $x = \pm r$.

$$\begin{aligned}
\int_{-r}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x &= \int_{-r}^{0} \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x + \int_{0}^{r} \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x \\
&= \lim_{A\to -r^+} \int_{A}^{0} \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x + \lim_{B\to r^-} \int_{0}^{B} \sqrt{\frac{r^2}{r^2-x^2}}\,\mathrm{d}x \\
&= \lim_{A\to -r^+} r\arcsin\frac{x}{r}\Big|_{x=A}^{0} + \lim_{B\to r^-} r\arcsin\frac{x}{r}\Big|_{x=0}^{B} \\
&= \lim_{A\to -r^+} \left(r\arcsin\frac{0}{r} - r\arcsin\frac{A}{r}\right) + \lim_{B\to r^-} \left(r\arcsin\frac{B}{r} - r\arcsin\frac{0}{r}\right) \\
&= r\arcsin 0 - r\arcsin(-1) + r\arcsin 1 - r\arcsin 0 \\
&= -r\cdot\left(-\frac{\pi}{2}\right) + r\cdot\frac{\pi}{2} \\
&= \pi r
\end{aligned}$$