Mathematics 1161: Midterm Exam 1 Study Guide Solutions

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1. Find the one-sided limit $\lim\limits_{x\to -2^+} \dfrac{x^2+5x+6}{x+2}$ as follows:

   (a) Does direct substitution apply?
   No, we would get the indeterminant form $\frac{0}{0}$.

   (b) Factoring the numerator and cancelling gives

   $$\lim_{x\to -2^+} \frac{x^2+5x+6}{x+2} = \lim_{x\to -2^+} \frac{(x+2)(x+3)}{x+2} = \lim_{x\to -2^+} (x+3)$$

   (c) Does direct substitution apply to the answer to part (b)?
   Yes.

   (d) Determine the limit.

   $$\lim_{x\to -2^+} \frac{x^2+5x+6}{x+2} = \lim_{x\to -2^+} (x+3) = -2+3 = 1$$

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2. Find the one-sided limits.

   (a) $\lim\limits_{x\to -2^+} \dfrac{x+2}{x^2+4x+4}$ Indeterminate form $\frac{0}{0}$: try algebra.

   $$= \lim_{x\to -2^+} \frac{x+2}{(x+2)(x+2)} \qquad \text{factoring}$$

   $$= \lim_{x\to -2^+} \frac{1}{x+2} \qquad \text{canceling}$$

   $$= \infty \qquad \text{As } x \to -2^+,\ x > -2. \text{ So, ``}\frac{1}{\text{small positive}} \to +\infty\text{''}$$

   (b) $\lim\limits_{x\to -2^-} \dfrac{x+2}{x^2+4x+4}$ Indeterminate form $\frac{0}{0}$: try algebra.

   $$= \lim_{x\to -2^-} \frac{x+2}{(x+2)(x+2)} \qquad \text{factoring}$$

   $$= \lim_{x\to -2^-} \frac{1}{x+2} \qquad \text{canceling}$$

   $$= -\infty \qquad \text{As } x \to -2^-,\ x < -2. \text{ So, ``}\frac{1}{\text{small negative}} \to -\infty\text{''}$$

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3. Treat m as a constant. Consider the function

   $$f(x) = \begin{cases} 4x+m & \text{if } x < 2 \\ -6x^2+2m & \text{if } x \geq 2. \end{cases}$$

   (a) Evaluate the limit in terms of m.

   $$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (-6x^2+2m) \qquad \text{As } x \to 2^+,\ x > 2. \text{ So, use } (-6x^2+2m) \text{ piece.}$$

   $$= -6(2)^2 + 2m \qquad \text{direct substitution}$$

   $$= -24 + 2m$$

(b) Evaluate the limit in terms of m.

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-}(4x+m) && \text{As } x\to 2^-,\ x<2. \text{ So, use } (4x+m) \text{ piece.}\\ &= 4(2)+m && \text{direct substitution}\\ &= 8+m\end{aligned}$$

(c) Find all values of m that make f continuous at $x = 2$ by determining the values of m such that $f(2)$, $\lim_{x\to 2^-} f(x)$, and $\lim_{x\to 2^+} f(x)$ are equal.

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^+} f(x) = f(2)\\ 8+m &= -24+2m = -24+2m\\ 32+m &= 2m\\ 32 &= m\end{aligned}$$

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4. Evaluate the limit.

$\lim_{x\to 3}\dfrac{x^2-5x+6}{x^2-7x+12}$ Indeterminate form $\frac{0}{0}$: try algebra.

$$\begin{aligned}&= \lim_{x\to 3}\frac{(x-2)(x-3)}{(x-4)(x-3)} && \text{factoring}\\ &= \lim_{x\to 3}\frac{x-2}{x-4} && \text{canceling}\\ &= \frac{3-2}{3-4} && \text{direct substitution}\\ &= -1\end{aligned}$$

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5. Evaluate the limit.

$\lim_{x\to 36}\dfrac{x^2-1296}{\sqrt{x}-6}$ Indeterminate form $\frac{0}{0}$: try algebra.

$$\begin{aligned}&= \lim_{x\to 36}\frac{(x-36)(x+36)}{\sqrt{x}-6} && \text{factoring}\\ &= \lim_{x\to 36}\frac{(x-36)(x+36)(\sqrt{x}+6)}{(\sqrt{x}-6)(\sqrt{x}+6)} && \text{multiplying by conjugate}\\ &= \lim_{x\to 36}\frac{(x-36)(x+36)(\sqrt{x}+6)}{x-36} && \text{using difference of squares formula}\\ &= \lim_{x\to 36}(x+36)(\sqrt{x}+6) && \text{canceling}\\ &= (36+36)(\sqrt{36}+6) && \text{direct substitution}\\ &= 864\end{aligned}$$

6. Evaluate the limit.

$\lim_{x\to 4}\left(\frac{1}{x-4}-\frac{8}{x^2-16}\right)$ Direct substution fails: try algebra.

$$
\begin{aligned}
&= \lim_{x\to 4}\left(\frac{1}{x-4}-\frac{8}{(x-4)(x+4)}\right) && \text{factoring}\\
&= \lim_{x\to 4}\left(\frac{x+4}{(x-4)(x+4)}-\frac{8}{(x-4)(x+4)}\right) && \text{common denominator}\\
&= \lim_{x\to 4}\frac{x+4-8}{(x-4)(x+4)} && \text{subtraction of fractions}\\
&= \lim_{x\to 4}\frac{x-4}{(x-4)(x+4)} && \\
&= \lim_{x\to 4}\frac{1}{x+4} && \text{canceling}\\
&= \frac{1}{4+4} && \text{direct substitution}\\
&= \frac{1}{8}
\end{aligned}
$$

7. Evaluate the limit.

$\lim_{x\to 0}\frac{x-3}{x^2(x-2)}=\infty$

As $x\to 0$, $x^2>0$. So, we have "$\frac{-3}{(\text{small positive})(-2)}=\frac{\text{positive}}{\text{small positive}}\to+\infty$".

8. Evaluate the limit.

$\lim_{x\to\infty}\frac{3-8x^4}{5+11x^4}$ Indeterminate form $\frac{-\infty}{\infty}$: try algebra. (Answer cannot be positive.)

$$
\begin{aligned}
&= \lim_{x\to\infty}\frac{\left(\frac{1}{x^4}\right)(3-8x^4)}{\left(\frac{1}{x^4}\right)(5+11x^4)} && \text{adjust by effective growth rate of denominator}\\
&= \lim_{x\to\infty}\frac{\frac{3}{x^4}-8}{\frac{5}{x^4}+11} && \text{distributing}\\
&= \frac{0-8}{0+11} && \text{As } x\to\infty,\ \frac{\text{constant}}{\text{positive power of } x}\to 0.\\
&= -\frac{8}{11}
\end{aligned}
$$

9. Evaluate the limit.

$\lim\limits_{x\to-\infty} \dfrac{2x^4-7x^2}{5x^6+8}$ Indeterminate form $\frac{\infty}{\infty}$: try algebra. (Answer cannot be negative.)

$$= \lim_{x\to-\infty} \frac{\left(\frac{1}{x^6}\right)(2x^4-7x^2)}{\left(\frac{1}{x^6}\right)(5x^6+8)}$$ adjust by effective growth rate of denominator

$$= \lim_{x\to-\infty} \frac{\frac{2}{x^2}-\frac{7}{x^4}}{5+\frac{8}{x^6}}$$ distribute

$$= \frac{0-0}{5+0}$$ As $x\to-\infty$, $\dfrac{\text{constant}}{\text{positive power of } x}\to 0.$

$$= 0$$

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10. Evaluate the limits.

(a) $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^6-6x^3-3}}{10x^3+10}$ Indeterminate form $\frac{\infty}{\infty}$: try algebra. (Answer cannot be negative.)

$$= \lim_{x\to\infty} \frac{\left(\frac{1}{x^3}\right)\sqrt{x^6-6x^3-3}}{\left(\frac{1}{x^3}\right)(10x^3+10)}$$ adjust by effective growth rate of denominator

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^6}}\sqrt{x^6-6x^3-3}}{\left(\frac{1}{x^3}\right)(10x^3+10)}$$ As $x\to\infty$, $\dfrac{1}{x^3}>0$. So, $\dfrac{1}{x^3}=\sqrt{\dfrac{1}{x^6}}$.

$$= \lim_{x\to\infty} \frac{\sqrt{\left(\frac{1}{x^6}\right)(x^6-6x^3-3)}}{\left(\frac{1}{x^3}\right)(10x^3+10)}$$ algebra rule: $a^nb^n=(ab)^n$

$$= \lim_{x\to\infty} \frac{\sqrt{1-\frac{6}{x^3}-\frac{3}{x^6}}}{10+\frac{10}{x^3}}$$ distribute

$$= \frac{\sqrt{1-0-0}}{10+0}$$ As $x\to\infty$, $\dfrac{\text{constant}}{\text{positive power of } x}\to 0.$

$$= \frac{1}{10}$$

(b) $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^6-6x^3-3}}{10x^3+10}$ Indeterminate form $\frac{\infty}{-\infty}$: try algebra. (Answer cannot be positive.)

$$= \lim_{x\to-\infty} \frac{\left(\frac{1}{x^3}\right)\sqrt{x^6-6x^3-3}}{\left(\frac{1}{x^3}\right)(10x^3+10)}$$ adjust by effective growth rate of denominator

$$= \lim_{x\to-\infty} \frac{-\sqrt{\frac{1}{x^6}}\sqrt{x^6-6x^3-3}}{\left(\frac{1}{x^3}\right)(10x^3+10)}$$ As $x\to-\infty$, $\dfrac{1}{x^3}<0$. So, $\dfrac{1}{x^3}=-\sqrt{\dfrac{1}{x^6}}$.

$$= \lim_{x\to-\infty} \frac{-\sqrt{\left(\frac{1}{x^6}\right)(x^6-6x^3-3)}}{\left(\frac{1}{x^3}\right)(10x^3+10)}$$ algebra rule: $a^nb^n=(ab)^n$

$$= \lim_{x\to-\infty} \frac{-\sqrt{1-\frac{6}{x^3}-\frac{3}{x^6}}}{10+\frac{10}{x^3}}$$ distribute

$$= \frac{-\sqrt{1-0-0}}{10+0}$$ As $x\to-\infty$, $\dfrac{\text{constant}}{\text{positive power of } x}\to 0.$

$$= -\frac{1}{10}$$

11. Evaluate the following limits.

(a) $\lim_{x\to\infty}\left(\sqrt{x^2-2x+1}-x\right)$ Indeterminate form $\infty-\infty$: try algebra.

Method 1: A method that works for these particular coefficients.

$$\begin{aligned}
&= \lim_{x\to\infty}\left(\sqrt{(x-1)^2}-x\right) && \text{factoring}\\
&= \lim_{x\to\infty}\left(|x-1|-x\right)\\
&= \lim_{x\to\infty}\left(x-1-x\right) && \text{As } x\to\infty,\ x-1>0. \text{ So, |x-1|=x-1.}\\
&= \lim_{x\to\infty} -1\\
&= -1
\end{aligned}$$

Method 2: A method that will work for many other choices of coefficients.

$$\begin{aligned}
&= \lim_{x\to\infty}\frac{\left(\sqrt{x^2-2x+1}-x\right)\left(\sqrt{x^2-2x+1}+x\right)}{\sqrt{x^2-2x+1}+x} && \text{multiply by conjugate}\\
&= \lim_{x\to\infty}\frac{x^2-2x+1-x^2}{\sqrt{x^2-2x+1}+x} && \text{difference of squares formula}\\
&= \lim_{x\to\infty}\frac{-2x+1}{\sqrt{x^2-2x+1}+x}\\
&= \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)(-2x+1)}{\left(\frac{1}{x}\right)\left(\sqrt{x^2-2x+1}+x\right)} && \text{adjust by effective growth rate of denominator}\\
&= \lim_{x\to\infty}\frac{-2+\frac{1}{x}}{\left(\frac{1}{x}\right)\sqrt{x^2-2x+1}+1} && \text{distribute}\\
&= \lim_{x\to\infty}\frac{-2+\frac{1}{x}}{\sqrt{\frac{1}{x^2}}\sqrt{x^2-2x+1}+1} && \text{As } x\to\infty,\ \frac{1}{x}>0. \text{ So, } \frac{1}{x}=\sqrt{\frac{1}{x^2}}.\\
&= \lim_{x\to\infty}\frac{-2+\frac{1}{x}}{\sqrt{\left(\frac{1}{x^2}\right)(x^2-2x+1)}+1} && \text{algebra rule: } a^nb^n=(ab)^n\\
&= \lim_{x\to\infty}\frac{-2+\frac{1}{x}}{\sqrt{1-\frac{2}{x}+\frac{1}{x^2}}+1} && \text{distribute}\\
&= \frac{-2+0}{\sqrt{1-0+0}+1} && \text{As } x\to\infty,\ \frac{\text{constant}}{\text{positive power of } x}\to 0.\\
&= -1
\end{aligned}$$

(b) $\lim_{x\to-\infty}\left(\sqrt{x^2-2x+1}-x\right)=+\infty$

"$\infty-(-\infty)=\infty+\infty=\infty$" (Not an indeterminate form.)

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12. A function is said to have a **horizontal asymptote** if either the limit at infinity exists or the limit at negative infinity exists. Show that each of the following functions has a horizontal asymptote by calculating the given limit.

(a) $\lim_{x\to\infty}\frac{-8x}{13+2x}=\lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)(-8x)}{\left(\frac{1}{x}\right)(13+2x)}=\lim_{x\to\infty}\frac{-8}{\frac{13}{x}+2}=\frac{-8}{0+2}=-4$

(b) $\lim_{x\to-\infty}\frac{11x-5}{x^3+8x-5}=\lim_{x\to-\infty}\frac{\left(\frac{1}{x^3}\right)(11x-5)}{\left(\frac{1}{x^3}\right)(x^3+8x-5)}=\lim_{x\to-\infty}\frac{\frac{11}{x^2}-\frac{5}{x^3}}{1+\frac{8}{x^2}-\frac{5}{x^3}}=\frac{0-0}{1+0-0}=0$

(c) $\lim_{x\to\infty} \frac{x^2-7x-13}{6-7x^2} = \lim_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)(x^2-7x-13)}{\left(\frac{1}{x^2}\right)(6-7x^2)} = \lim_{x\to\infty} \frac{1-\frac{7}{x}-\frac{13}{x^2}}{\frac{6}{x^2}-7} = \frac{1-0-0}{0-7} = -\frac{1}{7}$

(d) $\lim_{x\to\infty} \frac{\sqrt{x^2+8x}}{11-15x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)\sqrt{x^2+8x}}{\left(\frac{1}{x}\right)(11-15x)} = \lim_{x\to\infty} \frac{\sqrt{\left(\frac{1}{x^2}\right)(x^2+8x)}}{\left(\frac{1}{x}\right)(11-15x)} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{8}{x}}}{\frac{11}{x}-15} = \frac{\sqrt{1+0}}{0-15} = -\frac{1}{15}$

(e) $\lim_{x\to-\infty} \frac{\sqrt{x^2+8x}}{11-15x} = \lim_{x\to-\infty} \frac{\left(\frac{1}{x}\right)\sqrt{x^2+8x}}{\left(\frac{1}{x}\right)(11-15x)} = \lim_{x\to-\infty} \frac{-\sqrt{\left(\frac{1}{x^2}\right)(x^2+8x)}}{\left(\frac{1}{x}\right)(11-15x)} = \lim_{x\to-\infty} \frac{-\sqrt{1+\frac{8}{x}}}{\frac{11}{x}-15} = \frac{-\sqrt{1+0}}{0-15} = \frac{1}{15}$

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13. Find the values of c and d that make the following function

$$f(x) = \begin{cases} 3x & \text{if } x < 1, \\ cx^2 + d & \text{if } 1 \le x < 2 \\ 3x & \text{if } x \ge 2 \end{cases}$$

continuous for all x.

For $a < 1$, f is continous at a because $f(x) = 3x$ for all x near a.
For $1 < a < 2$, f is continous at a because $f(x) = cx^2 + d$ for all x near a.
For $a > 2$, f is continous at a because $f(x) = 3x$ for all x near a.

So, we must choose c and d such that f is continuous at 1 and at 2.

To make f continous at 1, we need $\lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$.

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3x = 3(1) = 3$
$f(1) = c(1)^2 + d = c + d$
$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} cx^2 + d = c(1)^2 + d = c + d$

So, we must have $3 = c + d$.

To make f continous at 2, we need $\lim_{x\to 2^-} f(x) = f(2) = \lim_{x\to 2^+} f(x)$.

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} cx^2 + d = c(2)^2 + d = 4c + d$
$f(2) = 3(2) = 6$
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 3x = 3(2) = 6$

So, we must have $6 = 4c + d$.

Solving the system of equations

$$\begin{cases} 3 = c + d \\ 6 = 4c + d \end{cases}$$

gives $c = 1$ and $d = 2$.

14. Use the **definition of derivative** to find $f'(2)$, where $f(x) = \sqrt{3+3x}$.

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} && \text{definition of derivative} \\
&= \lim_{h\to 0} \frac{\sqrt{3+3(2+h)} - \sqrt{3+3(2)}}{h} && \text{definition of } f \\
&= \lim_{h\to 0} \frac{\sqrt{3+6+3h} - \sqrt{3+6}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{9+3h} - \sqrt{9}}{h} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{9+3h} - \sqrt{9}\right)\left(\sqrt{9+3h} + \sqrt{9}\right)}{h\left(\sqrt{9+3h} + \sqrt{9}\right)} && \text{multiply by conjugate} \\
&= \lim_{h\to 0} \frac{9+3h-9}{h\left(\sqrt{9+3h} + \sqrt{9}\right)} && \text{difference of squares formula} \\
&= \lim_{h\to 0} \frac{3h}{h\left(\sqrt{9+3h} + \sqrt{9}\right)} \\
&= \lim_{h\to 0} \frac{3}{\left(\sqrt{9+3h} + \sqrt{9}\right)} && \text{canceling} \\
&= \frac{3}{\left(\sqrt{9+3(0)} + \sqrt{9}\right)} && \text{direct substitution} \\
&= \frac{1}{2}
\end{aligned}
$$

15. Use the **definition of derivative** to find $f'(x)$, where $f(x) = 2x + \frac{2}{x}$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{2(x+h) + \frac{2}{x+h} - \left(2x + \frac{2}{x}\right)}{h} \\
&= \lim_{h\to 0} \frac{2(x+h) + \frac{2}{x+h} - 2x - \frac{2}{x}}{h} \\
&= \lim_{h\to 0} \frac{2x + 2h + \frac{2}{x+h} - 2x - \frac{2}{x}}{h} \\
&= \lim_{h\to 0} \frac{2h + \frac{2}{x+h} - \frac{2}{x}}{h} \\
&= \lim_{h\to 0} \left[\frac{2h}{h} + \frac{1}{h}\left(\frac{2}{x+h} - \frac{2}{x}\right)\right] \\
&= \lim_{h\to 0} \left[2 + \frac{1}{h}\left(\frac{2x}{x(x+h)} - \frac{2(x+h)}{x(x+h)}\right)\right] \\
&= \lim_{h\to 0} \left[2 + \frac{1}{h}\left(\frac{2x}{x(x+h)} - \frac{2x+2h}{x(x+h)}\right)\right] \\
&= \lim_{h\to 0} \left[2 + \frac{1}{h}\left(\frac{2x - (2x+2h)}{x(x+h)}\right)\right] \\
&= \lim_{h\to 0} \left[2 + \frac{1}{h}\left(\frac{2x - 2x - 2h}{x(x+h)}\right)\right] \\
&= \lim_{h\to 0} \left[2 + \frac{1}{h}\left(\frac{-2h}{x(x+h)}\right)\right] \\
&= \lim_{h\to 0} \left[2 + \frac{-2}{x(x+h)}\right] \\
&= 2 + \frac{-2}{x(x+0)} \\
&= 2 - \frac{2}{x^2}
\end{aligned}$$

Use this to find the equation of the tangent line to the graph of $y = 2x + \frac{2}{x}$ at the point $(-4, -8.5)$.

Point is $(x_0, y_0) = (-4, -\frac{17}{2})$. Slope is $m = f'(-4) = 2 - \frac{2}{(-4)^2} = 2 - \frac{1}{8} = \frac{15}{8}$.

$$\begin{aligned}
y - y_0 &= m(x - x_0) \\
y - (-\tfrac{17}{2}) &= \tfrac{15}{8}\big(x - (-4)\big) \\
y + \tfrac{17}{2} &= \tfrac{15}{8}(x + 4)
\end{aligned}$$

16. Differentiate $f(x) = 5x^2 - 2x - 18$.

$$\begin{aligned} f(x) &= 5x^2 - 2x - 18 \\ f'(x) &= 5(2x) - 2(1) - 0 \\ &= 10x - 2 \end{aligned}$$

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17. If $f(x) = \dfrac{3x}{1+x^2}$, find $f'(5)$.

$$\begin{aligned} f(x) &= \frac{3x}{1+x^2} \\ f'(x) &= \frac{(1+x^2)\frac{d}{dx}[3x] - (3x)\frac{d}{dx}[1+x^2]}{(1+x^2)^2} \\ &= \frac{(1+x^2)(3) - (3x)(2x)}{(1+x^2)^2} \\ &= \frac{3-3x^2}{(1+x^2)^2} \\ f'(5) &= \frac{3-3(5)^2}{(1+(5)^2)^2} \\ &= \frac{-72}{26^2} \\ &= -\frac{18}{169} \end{aligned}$$

Use this to find the equation of the tangent line to the curve $y = \frac{3x}{1+x^2}$ at the point $(5, \frac{15}{26})$.

Point is $(x_0, y_0) = (5, \frac{15}{26})$. Slope is $m = f'(5) = -\frac{18}{169}$.

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - (\tfrac{15}{26}) &= -\tfrac{18}{169}\big(x - (5)\big) \\ y - \tfrac{15}{26} &= -\tfrac{18}{169}\big(x - 5\big) \end{aligned}$$

18. For which values of x does the graph of

$$f(x) = 4x^3 - 18x^2 - 36$$

have a horizontal tangent?

Want to find x such that $f'(x) = 0$.

$$\begin{aligned} 0 &= f'(x) \\ &= 4(3x^2) - 18(2x) - 0 \\ &= 12x^2 - 36x \\ &= 12x(x-3) \end{aligned}$$

Hence,

$$x = 0 \qquad \text{and} \qquad \begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

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19. Differentiate $f(x) = x^3 + 5e^{3x}$.

$$\begin{aligned} f(x) &= x^3 + 5e^{3x} \\ f'(x) &= 3x^2 + 5(3e^{3x}) \\ &= 3x^2 + 15e^{3x} \end{aligned}$$

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20. Differentiate $f(x) = \dfrac{9x}{3x-7}$.

$$\begin{aligned} f(x) &= \frac{9x}{3x-7} \\ f'(x) &= \frac{(3x-7)\frac{d}{dx}[9x] - (9x)\frac{d}{dx}[3x-7]}{(3x-7)^2} \\ &= \frac{(3x-7)(9) - (9x)(3)}{(3x-7)^2} \\ &= \frac{-63}{(3x-7)^2} \end{aligned}$$

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21. Differentiate $g(x) = (3x^2 - 4x - 5)e^x$.

$$\begin{aligned} g(x) &= (3x^2 - 4x - 5)e^x \\ g'(x) &= \tfrac{d}{dx}[3x^2 - 4x - 5]e^x + (3x^2 - 4x - 5)\tfrac{d}{dx}[e^x] \\ &= (6x - 4)e^x + (3x^2 - 4x - 5)e^x \\ &= (3x^2 + 2x - 9)e^x \end{aligned}$$

22. Differentiate $f(x) = \dfrac{xe^x + 5}{x^2 + 12}$.

$$\begin{aligned} f(x) &= \frac{xe^x + 5}{x^2 + 12} \\ f'(x) &= \frac{(x^2+12)\frac{d}{dx}[xe^x+5] - (xe^x+5)\frac{d}{dx}[x^2+12]}{(x^2+12)^2} \\ &= \frac{(x^2+12)\left(\frac{d}{dx}[x](e^x) + (x)\frac{d}{dx}[e^x] + 0\right) - (xe^x+5)(2x+0)}{(x^2+12)^2} \\ &= \frac{(x^2+12)\left((1)(e^x) + (x)(e^x)\right) - (xe^x+5)(2x)}{(x^2+12)^2} \\ &= \frac{(x^2+12)(e^x + xe^x) - (xe^x+5)(2x)}{(x^2+12)^2} \end{aligned}$$

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23. Differentiate $f(x) = 5\sin x + 2\cos x$.

$$\begin{aligned} f(x) &= 5\sin x + 2\cos x \\ f'(x) &= 5(\cos x) + 2(-\sin x) \\ &= 5\cos x - 2\sin x \end{aligned}$$

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24. Differentiate $f(x) = x^4\cos x$.

$$\begin{aligned} f(x) &= x^4\cos x \\ f'(x) &= \tfrac{d}{dx}[x^4](\cos x) + (x^4)\tfrac{d}{dx}[\cos x] \\ &= (4x^3)(\cos x) + (x^4)(-\sin x) \\ &= 4x^3\cos x - x^4\sin x \end{aligned}$$

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25. Differentiate $f(x) = 2\sec x + 4e^x\tan x$.

$$\begin{aligned} f(x) &= 2\sec x + 4e^x\tan x \\ f'(x) &= 2(\sec x\tan x) + 4\left(e^x\sec^2 x + e^x\tan x\right) \\ &= 2\sec x\tan x + 4e^x\sec^2 x + 4e^x\tan x \end{aligned}$$

26. Evaluate the limit.

$$\lim_{x\to 0} \frac{\sin 8x}{\sin 3x}$$

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 8x}{\sin 3x} &= \lim_{x\to 0} \frac{(8x)\left(\frac{\sin(8x)}{8x}\right)}{(3x)\left(\frac{\sin(3x)}{3x}\right)} \\ &= \lim_{x\to 0} \frac{8\left(\frac{\sin(8x)}{8x}\right)}{3\left(\frac{\sin(3x)}{3x}\right)} \\ &= \frac{8(1)}{3(1)} \\ &= \frac{8}{3}\end{aligned}$$

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27. Evaluate the limit.

$$\lim_{\theta\to 0} \frac{\sin(7\theta)}{\tan(6\theta)}$$

$$\begin{aligned}\lim_{\theta\to 0} \frac{\sin(7\theta)}{\tan(6\theta)} &= \lim_{\theta\to 0} \frac{(7\theta)\left(\frac{\sin(7\theta)}{7\theta}\right)}{(6\theta)\left(\frac{\tan(6\theta)}{6\theta}\right)} \\ &= \lim_{\theta\to 0} \frac{7\left(\frac{\sin(7\theta)}{7\theta}\right)}{6\left(\frac{\tan(6\theta)}{6\theta}\right)} \\ &= \frac{7(1)}{6(1)} \\ &= \frac{7}{6}\end{aligned}$$

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28. If a ball is thrown straight up into the air with an initial velocity of 50 ft/s, its height in feet after t seconds is given by $h(t) = 50t - 16t^2$. Find the average velocity for the time period beginning when $t = 2$ and lasting

(a) 0.5 seconds

$$\begin{aligned}\text{avg. vel.} &= \frac{h(2.5) - h(2)}{2.5 - 2} \\ &= \frac{25 - 36}{0.5} \\ &= -22 \text{ ft/s}\end{aligned}$$

(b) 0.1 seconds

$$\begin{aligned}\text{avg. vel.} &= \frac{h(2.1) - h(2)}{2.1 - 2} \\ &= \frac{34.44 - 36}{0.1} \\ &= -15.6 \text{ ft/s}\end{aligned}$$

Calculate the instantaneous velocity of the ball when $t = 2$.

$$\begin{aligned} h(t) &= 50t - 16t^2 \\ h'(t) &= 50(1) - 16(2t) \\ &= 50 - 32t \\ h'(2) &= 50 - 32(2) \\ &= -14 \text{ ft/s} \end{aligned}$$

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29. The displacement (in meters) of a particle moving in a straight line is given by

$$s = t^2 - 7t + 16,$$

where t is measured in seconds.

(a) Find the average velocity over the time interval [3,4].

$$\begin{aligned} \text{avg. vel.} &= \frac{s(4) - s(3)}{4 - 3} \\ &= \frac{4 - 4}{1} \\ &= 0 \text{ m/s} \end{aligned}$$

(b) Find the average velocity over the time interval [4,5].

$$\begin{aligned} \text{avg. vel.} &= \frac{s(5) - s(4)}{5 - 4} \\ &= \frac{6 - 4}{1} \\ &= 2 \text{ m/s} \end{aligned}$$

(c) Find the instantaneous velocity when $t = 4$.

$$\begin{aligned} s(t) &= t^2 - 7t + 16 \\ s'(t) &= 2t - 7 \\ s'(4) &= 2(4) - 7 \\ &= 1 \text{ m/s} \end{aligned}$$

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30. Let $f(x) = e^{\sqrt{3x+8}}$. Find $f'(x)$.

$$\begin{aligned} f'(x) &= e^{(\sqrt{3x+8})} \tfrac{d}{dx}\Big(\sqrt{3x+8}\Big) && \text{Chain Rule} \\ &= e^{\sqrt{3x+8}} \tfrac{d}{dx}\Big((3x+8)^{\frac{1}{2}}\Big) \\ &= e^{\sqrt{3x+8}} \cdot \tfrac{1}{2}\Big(3x+8\Big)^{-\frac{1}{2}} \tfrac{d}{dx}\Big(3x+8\Big) && \text{Chain Rule} \\ &= e^{\sqrt{3x+8}} \cdot \tfrac{1}{2}(3x+8)^{-\frac{1}{2}} \cdot 3 \end{aligned}$$

31. Let $f(x) = -9e^{x\cos x}$. Find $f'(x)$.

$$\begin{aligned} f'(x) &= -9e^{(x\cos x)}\,\tfrac{d}{dx}\Big(x\cos x\Big) && \text{Chain Rule}\\ &= -9e^{x\cos x}\big(\cos x + x(-\sin x)\big) && \text{Product Rule}\\ &= -9e^{x\cos x}\big(\cos x - x\sin x\big) \end{aligned}$$

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32. Let $f(x) = \sqrt{\sin(e^{x^2\sin(x)})}$. Find $f'(x)$.

$$\begin{aligned} f(x) &= \big(\sin(e^{x^2\sin x})\big)^{\frac{1}{2}}\\ f'(x) &= \tfrac{1}{2}\Big(\sin(e^{x^2\sin x})\Big)^{-\frac{1}{2}}\tfrac{d}{dx}\Big(\sin(e^{x^2\sin x})\Big) && \text{Chain Rule}\\ &= \tfrac{1}{2}\big(\sin(e^{x^2\sin x})\big)^{-\frac{1}{2}}\cos(e^{x^2\sin x})\,\tfrac{d}{dx}\Big(e^{x^2\sin x}\Big) && \text{Chain Rule}\\ &= \tfrac{1}{2}\big(\sin(e^{x^2\sin x})\big)^{-\frac{1}{2}}\cos(e^{x^2\sin x})\,e^{(x^2\sin x)}\,\tfrac{d}{dx}\Big(x^2\sin x\Big) && \text{Chain Rule}\\ &= \tfrac{1}{2}\big(\sin(e^{x^2\sin x})\big)^{-\frac{1}{2}}\cos(e^{x^2\sin x})\,e^{x^2\sin x}\,(2x\sin x + x^2\cos x) && \text{Product Rule} \end{aligned}$$

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33. Use the ε-δ definition of a limit to prove that $\lim_{x\to -3} f(x) = 1$, where $f(x) = 2x^2 + 12x + 19$.

Proof. Let $\varepsilon > 0$ be given, and assume $0 < \left|x - (-3)\right| < \delta$, where $\delta = \sqrt{\frac{\varepsilon}{2}}$. So, we are assuming that $0 < |x+3| < \delta$. Observe that

$$\begin{aligned} \left|f(x) - 1\right| &= \left|(2x^2 + 12x + 19) - 1\right|\\ &= \left|2x^2 + 12x + 18\right|\\ &= \left|2(x^2 + 6x + 9)\right|\\ &= 2\left|x^2 + 6x + 9\right|\\ &= 2\left|(x+3)^2\right|\\ &= 2\,|x+3|^2\\ &< 2\delta^2\\ &= 2\left(\sqrt{\tfrac{\varepsilon}{2}}\right)^2\\ &= 2\cdot\tfrac{\varepsilon}{2}\\ &= \varepsilon \end{aligned}$$

We have shown $\left|f(x) - 1\right| < \varepsilon$. So, it follows from the definition of limit that $\lim_{x\to -3} f(x) = 1$. □

Given $\varepsilon > 0$, list three different choices of δ that ensure $\left|f(x) - 1\right| < \varepsilon$ whenever $0 < \left|x - (-3)\right| < \delta$.

We showed above that $\delta = \sqrt{\frac{\varepsilon}{2}}$ has the desired property, and we so any smaller (positive) choice of δ will have the desired property, too. In particular, we could choose $\delta = \frac{1}{2}\sqrt{\frac{\varepsilon}{2}}$ or $\delta = \frac{1}{4}\sqrt{\frac{\varepsilon}{2}}$.