Practice Midterm 1 Solutions – Math 1161.0X

1. Decide if the following statements are TRUE or FALSE. **You do NOT need to justify your answers.**

(a) (1 point) If f and g are differentiable at a then $f \circ g$ is differentiable at a.

Solution: **F** (f must be differentiable at $g(a)$ to apply the chain rule here.)

(b) (1 point) $\lim_{x\to\infty} \arctan x = 1$.

Solution: **F** ($\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$)

(c) (1 point) If $f'(c) = 0$ and $g'(c) = 0$ then $\frac{\mathrm{d}}{\mathrm{d}x}(f(x)g(x))|_{x=c} = 0$.

Solution: **T** (By the product rule $(f \cdot g)'(c) = f'(c)g(c) + f(c)g'(c) = 0 \cdot g(c) + f(c) \cdot 0 = 0$)

(d) (1 point) If $f(0) = 10$, $f(7) = 4$, and f is continuous on the closed interval $[-1, 8]$ then there must be some number $c \in (-1, 8)$ such that $f(c) = 9$.

Solution: **T** (This follows from the Intermediate Value Theorem.)

(e) (1 point) If f' has a vertical asymptote at c then f must have a vertical asymptote at c.

Solution: **F** (f might only have a vertical tangent at c.)

(f) (1 point) If f is left continuous at c and right continuous at c then f is continuous at c.

Solution: **T** (If $\lim_{x\to c^-} f(x) = f(c)$ and $\lim_{x\to c^+} f(x) = f(c)$ then $\lim_{x\to c} f(x) = f(c)$.)

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

(a) (2 points) Give an example of a function $f(x)$ which is continuous on the interval $(-\infty, 1]$ and continuous on the interval $(1, \infty)$ but **not continuous** on the interval $(-\infty, \infty)$.

Solution:

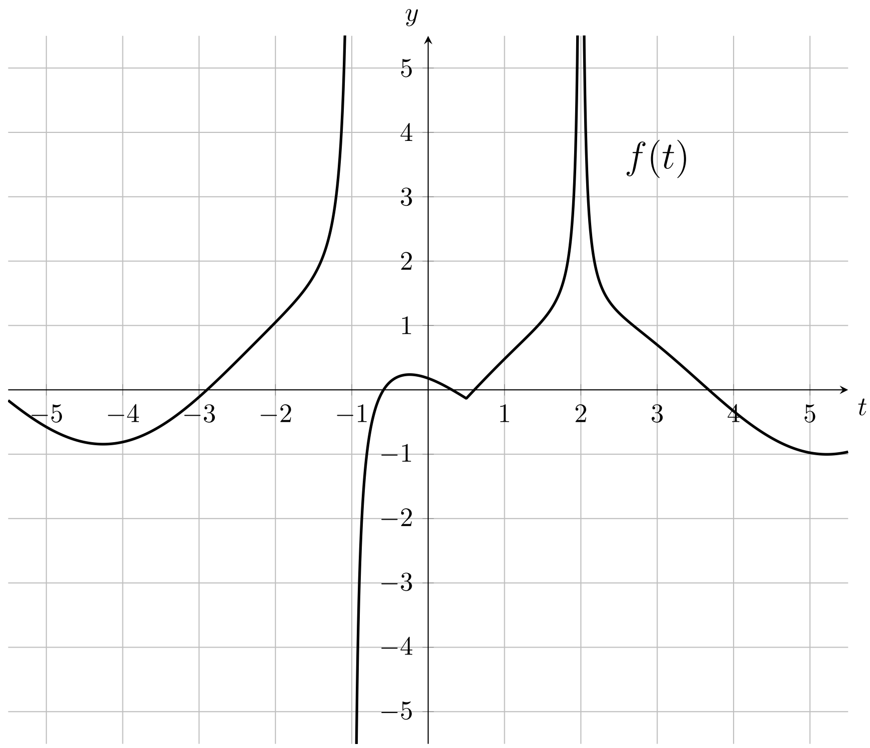
$$f(x) = \begin{cases} 2, & x \le 1 \\ 3, & x > 1 \end{cases}$$

(b) (2 points) Give an example of a function $f(x)$ with exactly one vertical asymptote and two distinct horizontal asymptotes.

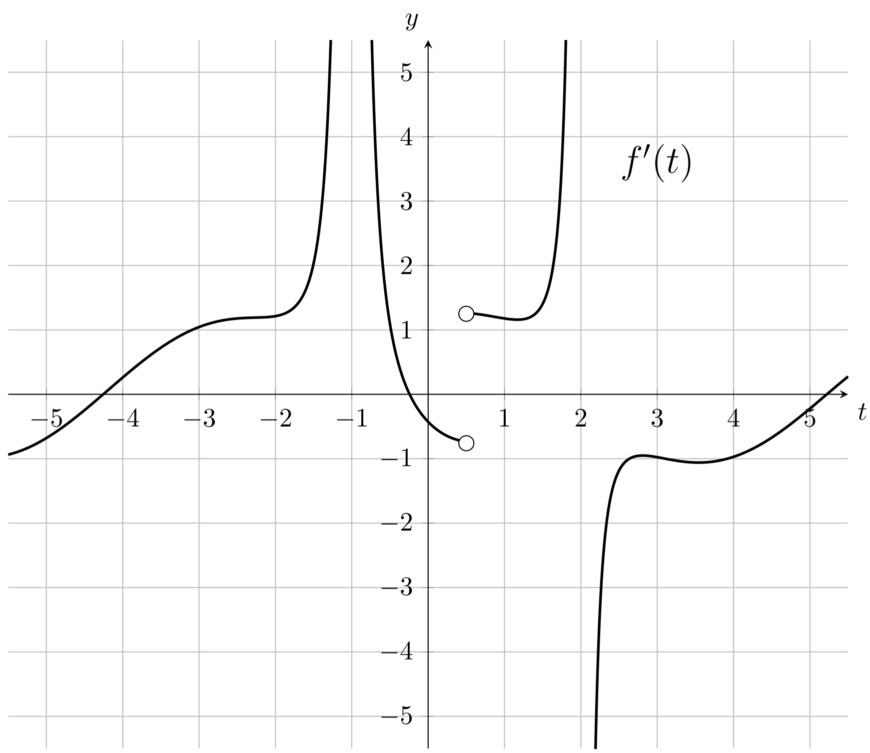
Solution:

$$f(x) = \frac{1}{x} + \arctan x$$

3. (5 points) The graph of a function $y = f(t)$ is given below. Sketch its derivative.



Solution:



4. Let

$$f(x) = \frac{1}{2x}.$$

(a) (3 points) What is $f'(x)$?

Solution:

$$f'(x) = -\frac{1}{2x^2}$$

(b) (7 points) Using **only the definition of the derivative** compute $f'(x)$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\frac{1}{2(x+h)} - \frac{1}{2x}}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{x-(x+h)}{2x(x+h)}\right)}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{-h}{2x(x+h)}\right)}{h} \\ &= \lim_{h \to 0} \frac{-1}{2x(x+h)} \\ &= -\frac{1}{2x^2} \end{aligned}$$

5. Find $\frac{dy}{dx}$ for the following functions. You do not need to simplify your answers.

(a) (5 points) $y = (10 - e)\sqrt[5]{x^7} - \dfrac{4}{\sqrt[3]{\pi + 4}} + e^2$

Solution:

$$\frac{dy}{dx} = \frac{7}{5}(10 - e)x^{\frac{2}{5}}$$

(b) (5 points) $y = \dfrac{4e^x + 6x^2 - 3 + \tan x}{11x - 4x^2}$

Solution:

$$\frac{dy}{dx} = \frac{(4e^x + 12x + \sec^2 x)(11x - 4x^2) - (4e^x + 6x^2 - 3 + \tan x)(11 - 8x)}{(11x - 4x^2)^2}$$

(c) (5 points) $y = (\sin x)(e^x)(x^7)$

Solution:

$$\frac{dy}{dx} = (\sin x)(e^x)(7x^6) + (\sin x)(e^x)(x^7) + (\cos x)(e^x)(x^7)$$

6. (5 points) Show that

$$\lim_{x\to -2} -\tfrac{1}{4}x^3 + 1 = 3$$

using **only the limit laws**.

Solution:

$$\begin{aligned}
\lim_{x\to -2}\left(-\tfrac{1}{4}x^3+1\right) &= \lim_{x\to -2} -\tfrac{1}{4}x^3 + \lim_{x\to -2} 1\\
&= -\tfrac{1}{4}\left(\lim_{x\to -2} x\right)^3 + 1\\
&= -\tfrac{1}{4}(-2)^3 + 1\\
&= 3
\end{aligned}$$

7. Evaluate the following limits using any technique you like.

(a) (5 points) $\lim_{x\to 0} \frac{3x^2+7x}{2x+5}$

Solution:

$$\lim_{x\to 0} \frac{3x^2+7x}{2x+5} = \frac{3\cdot 0^2 + 7\cdot 0}{2\cdot 0 + 5} = 0$$

(b) (5 points) $\lim_{x\to\infty} \frac{\sqrt{x^2+\sqrt[3]{2x^6+1}}}{x+5}$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+\sqrt[3]{2x^6+1}}}{x+5} &= \lim_{x\to\infty} \frac{\sqrt{x^2+\sqrt[3]{2x^6+1}}}{x+5}\cdot\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}\sqrt[3]{2x^6+1}}}{\frac{x}{x}+\frac{5}{x}}\\
&= \lim_{x\to\infty} \frac{\sqrt{1+\sqrt[3]{\frac{2x^6}{x^6}+\frac{1}{x^6}}}}{1+\frac{5}{x}}\\
&= \lim_{x\to\infty} \frac{\sqrt{1+\sqrt[3]{2+\frac{1}{x^6}}}}{1+\frac{5}{x}}\\
&= \frac{\sqrt{1+\sqrt[3]{2+0}}}{1+0}\\
&= \sqrt{1+\sqrt[3]{2}}
\end{aligned}$$

8. Consider the function

$$g(t) = \frac{2t^3-t^2}{t^2-t-2}$$

(a) (5 points) Find all vertical asymptotes of g

Solution:

$$\begin{aligned} g(t) &= \frac{2t^3 - t^2}{t^2 - t - 2} \\ &= \frac{(2t-1)t^2}{(t-2)(t+1)} \end{aligned}$$

The vertical asymptotes are $t = 2$ and $t = -1$.

(b) (5 points) Find all horizontal asymptotes of g.

Solution:

$$\begin{aligned} \lim_{t\to\infty} g(t) &= \lim_{t\to\infty} \frac{2t^3 - t^2}{t^2 - t - 2} \\ &= \lim_{t\to\infty} \frac{2t^3 - t^2}{t^2 - t - 2} \cdot \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \\ &= \lim_{t\to\infty} \frac{2t - 1}{1 - \frac{1}{t} - \frac{2}{t^2}} \\ &= \infty \end{aligned}$$

$$\begin{aligned} \lim_{t\to-\infty} g(t) &= \lim_{t\to-\infty} \frac{2t^3 - t^2}{t^2 - t - 2} \\ &= \lim_{t\to-\infty} \frac{2t^3 - t^2}{t^2 - t - 2} \cdot \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{1}{t^2}\right)} \\ &= \lim_{t\to-\infty} \frac{2t - 1}{1 - \frac{1}{t} - \frac{2}{t^2}} \\ &= -\infty \end{aligned}$$

Hence g has no horizontal asymptotes

(c) (5 points) Find the equation for the tangent line to the graph of g at the point $t = 1$.

Solution:

$$\begin{aligned} g'(t) &= \frac{(6t^2 - 2t)(t^2 - t - 2) - (2t^3 - t^2)(2t - 1)}{(t^2 - t - 2)^2} \\ g'(1) &= \frac{(6 \cdot 1^2 - 2 \cdot 1)(1^2 - 1 - 2) - (2 \cdot 1^3 - 1^2)(2 \cdot 1 - 1)}{(1^2 - 1 - 2)^2} \\ &= \frac{(4)(-2) - (1)(1)}{(-2)^2} \\ &= -\tfrac{9}{4} \\ g(t) &= \frac{2t^3 - t^2}{t^2 - t - 2} \end{aligned}$$

$$g(1) = \frac{2 \cdot 1^3 - 1^2}{1^2 - 1 - 2} = -\tfrac{1}{2}$$

Thus the tangent line to the graph of g at the point $t = 1$ has equation

$$y = -\tfrac{9}{4}(t-1) - \tfrac{1}{2}$$

9. A particle's position at time t is given by the equation

$$s(t) = \cos(p\pi t).$$

(a) (5 points) Find the average velocity of the particle on the time interval from $t = 0$ to $t = \frac{2}{p}$

Solution:

$$\begin{aligned}
\text{average velocity} &= \frac{\text{change in position}}{\text{change in time}} \\
&= \frac{s\left(\frac{2}{p}\right) - s(0)}{\frac{2}{p} - 0} \\
&= \frac{\cos\left(p \cdot \pi \cdot \frac{2}{p}\right) - \cos(p \cdot \pi \cdot 0)}{\frac{2}{p}} \\
&= \frac{\cos(2\pi) - \cos(0)}{\frac{2}{p}} \\
&= \frac{1-1}{\frac{2}{p}} \\
&= 0
\end{aligned}$$

(b) (5 points) Give an equation for the acceleration of the particle at time t.

Solution:

$$\begin{aligned}
a(t) &= \tfrac{\mathsf{d}^2}{\mathsf{d}t^2} \cos(p\pi t) \\
&= \tfrac{\mathsf{d}}{\mathsf{d}t} \left(-p\pi \sin(p\pi t)\right) \\
&= -p^2\pi^2 \cos(p\pi t)
\end{aligned}$$

10. Simplify the following expressions

(a) (5 points) $\sec(\tan^{-1} x)$.

Solution:

$$\sec(\tan^{-1} x) = \sqrt{1 + [\tan(\tan^{-1} x)]^2}$$

$$= \sqrt{1 + x^2}$$

(b) (5 points) $\sin(\cos^{-1} x)$

Solution:

$$\begin{aligned} \sin(\cos^{-1} x) &= \sqrt{1 - [\cos(\cos^{-1} x)]^2} \\ &= \sqrt{1 - x^2} \end{aligned}$$