Practice Midterm 2 – Math 1161.0X

1. Decide if the following statements are TRUE or FALSE. **You do NOT need to justify your answers.**
   (a) (1 point) If $f'(c) = 0$ then f must have a local minimum or a local maximum at c.

   Solution: **F** (Not all critical points are local extrema.)

   (b) (1 point) If f is continuous on the closed interval $[0, 7]$ then f has a global maximum and a global mininum on $[0, 7]$.

   Solution: **T** (Extreme Value Theorem)

   (c) (1 point) $f(x) = x^3$ has both a critical point and an inflection point at $x = 0$.

   Solution: **T** ($f'(0) = 0$ and concavity of f switches at $x = 0$.)

   (d) (1 point) If f is nondecreasing on the interval $[2, 5]$ and nondecreasing on the interval $[5, 10]$ then it is nondecreasing on the interval $[2, 10]$.

   Solution: **T** (This can be proven from the definition of nondecreasing.)

   (e) (1 point) If f is an even function then f has a critical point at $x = 0$.

   Solution: **F** (This statement would be true if we had the addtional assumption that 0 is in the domain of f. In that case we could argue that if f is even and differentiable at x then $f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx} f(-x) = -f'(-x)$. In particular $f'(0) = -f'(0)$ so $f'(0) = 0$ or $f'(0)$ does not exist.)

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**
   (a) (2 points) Give an example of a function $f(x)$ with domain $(-\infty, \infty)$ which is both odd and even.

   Solution:
   $$f(x) = 0$$

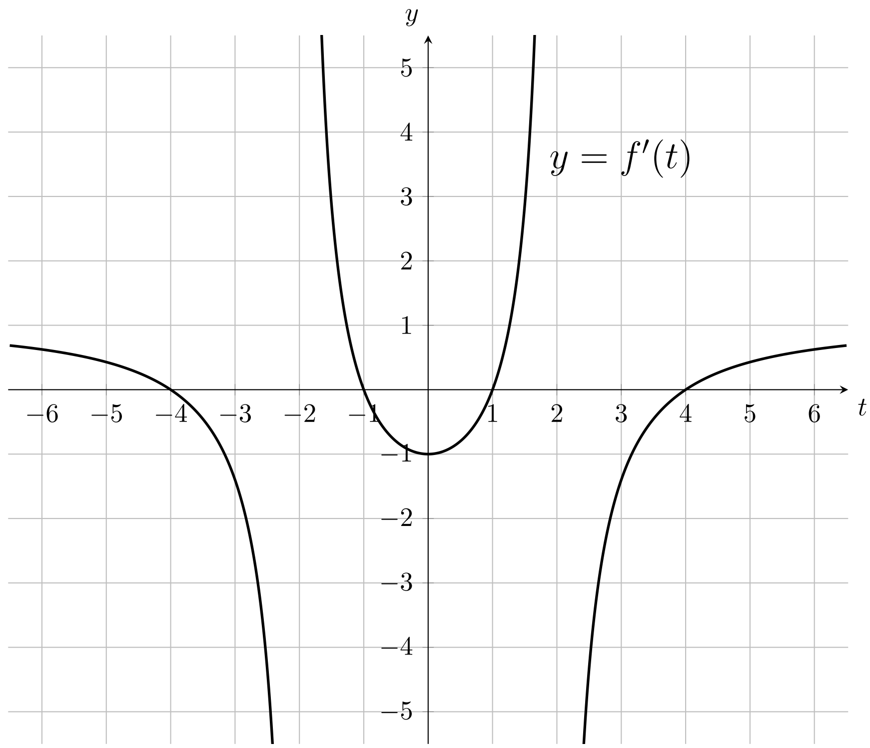
   (b) (2 points) Give an example of a continuous function $f(x)$ with domain $(-\infty, \infty)$ which has no local extrema.

   Solution:
   $$f(x) = x$$

   (c) (2 points) Give an example of a function $f(x)$ which is decreasing and concave up on the interval $(-\infty, \infty)$.

   Solution:
   $$f(x) = e^{-x}$$

3. The graph the **derivative** $y = f'(t)$ is given below.



(a) (3 points) At what t-values does the function f whose **derivative** is pictured above have critical points?

Solution: $t = -4, t = -2, t = 2, t = 4$

(b) (3 points) What are the intervals of increase for the function f?

Solution: $(-\infty, -4]$, $(-2, -1]$, $[1, 2)$, $[4, \infty)$

(c) (3 points) What are the intervals of decrease for f?

Solution: $[-4, -2)$, $[-1, 1]$, $(2, 4]$

(d) (3 points) At what t-values does f have an inflection point.

Solution: $t = 0$

(e) (3 points) On what open intervals is f concave up?

Solution: $(0, 2)$, $(2, \infty)$

(f) (3 points) On what open intervals is f concave down?

Solution: $(-\infty, -2)$, $(-2, 0)$

4. Consider the curve C in the xy-plane satisfying the equation

$$x^3y - x^2y^2 + xy^3 = -10$$

(a) (3 points) Compute $\frac{dy}{dx}$ in terms of x and y.

Solution:

$$\begin{aligned}
\tfrac{d}{dx}\left(x^3y - x^2y^2 + xy^3\right) &= \tfrac{d}{dx}(-10)\\
3x^2y + x^3\tfrac{dy}{dx} - 2xy^2 - 2x^2y\tfrac{dy}{dx} + y^3 + 3xy^2\tfrac{dy}{dx} &= 0\\
x^3\tfrac{dy}{dx} - 2x^2y\tfrac{dy}{dx} + 3xy^2\tfrac{dy}{dx} &= -3x^2y + 2xy^2 - y^3\\
\tfrac{dy}{dx}(x^3 - 2x^2y + 3xy^2) &= -3x^2y + 2xy^2 - y^3\\
\tfrac{dy}{dx} &= \frac{-3x^2y + 2xy^2 - y^3}{x^3 - 2x^2y + 3xy^2}
\end{aligned}$$

(b) (3 points) What is the equation of the tangent line to the curve C at the point $(2, -1)$?

Solution: If $x = 2$ and $y = -1$ then

$$\begin{aligned}
\tfrac{dy}{dx} &= \frac{-3\cdot 2^2\cdot(-1) + 2\cdot 2\cdot(-1)^2 - (-1)^3}{2^3 - 2\cdot 2^2\cdot(-1) + 3\cdot 2\cdot(-1)^2}\\
&= \tfrac{12+4+1}{8+8+6}\\
&= \tfrac{17}{22}
\end{aligned}$$

Thus the tangent line to the curve C at the point $(2, -1)$ has equation

$$y = \tfrac{17}{22}(x-2) + 1.$$

5. (5 points) Find $\frac{dy}{dx}$ if $y = \dfrac{(x-5)^2\tan^4 x}{(\cos x + 2)(x^4 + 10)}$. You **do not need to simplify** your answer.

Solution:

$$\begin{aligned}
\ln|y| &= \ln\left(\frac{|x-5|^2|\tan^4 x|}{|\cos x + 2||x^4 + 10|}\right)\\
&= 2\ln|x-5| + 4\ln|\tan x| - \ln|\cos x + 2| - \ln|x^4 + 10|
\end{aligned}$$

So

$$\begin{aligned}
\tfrac{d}{dx}\ln|y| &= \tfrac{d}{dx}\left(2\ln|x-5| + 4\ln|\tan x| - \ln|\cos x + 2| - \ln|x^4 + 10|\right)\\
\frac{y'}{y} &= \frac{2}{x-5} + \frac{4\sec^2 x}{\tan x} + \frac{\sin x}{\cos x + 2} - \frac{4x^3}{x^4 + 10}\\
y' &= y\left(\frac{2}{x-5} + \frac{4\sec^2 x}{\tan x} + \frac{\sin x}{\cos x + 2} - \frac{4x^3}{x^4 + 10}\right)\\
y' &= \frac{(x-5)^2\tan^4 x}{(\cos x + 2)(x^4 + 10)}\left(\frac{2}{x-5} + \frac{4\sec^2 x}{\tan x} + \frac{\sin x}{\cos x + 2} - \frac{4x^3}{x^4 + 10}\right)
\end{aligned}$$

6. Let

$$f(x) = \arctan(x^2)$$

(a) (3 points) What is the domain of f?

Solution: f is a composition of continuous functions so the domain of f is $(-\infty, \infty)$.

(b) (3 points) Is f even, odd, periodic?

Solution: $f(-x) = \arctan((-x)^2) = \arctan(x^2) = f(x)$ so f is even. It is not odd or periodic.

(c) (3 points) Compute $f'(x)$.

Solution:

$$f'(x) = \frac{2x}{1 + (x^2)^2} = \frac{2x}{1 + x^4}$$

(d) (5 points) What are x-coordinates for the critical points of f?

Solution: Critical points from the numerator:

$$0 = 2x$$
$$x = 0$$

Critical points from the denominator:

$$0 = 1 + x^4$$

no solutions

$\boxed{x = 0}$

(e) (5 points) What are the intervals of increase for f?

Solution: $(1 + x^4) > 0$ for all x so $f'(x)$ is positive if and only if $2x > 0$.
$[0, \infty)$ is the interval of increase.

(f) (5 points) What are the intervals of decrease for f?

Solution: $f'(x)$ is negative if and only if $2x < 0$.
$(-\infty, 0]$ is the interval of decrease.

(g) (5 points) What is the second derivative of f?

Solution:

$$f''(x) = \frac{2(1+x^4) - 2x(4x^3)}{(1+x^4)^2} = \frac{2+2x^4-8x^4}{(1+x^4)^2} = \frac{2-6x^4}{(1+x^4)^2}$$

(h) (5 points) On what open intervals is f concave up?

Solution: $(1+x^4)^2 > 0$ for all x so $f''(x)$ is positive if and only if $2-6x^4 > 0$.

$$\begin{aligned} 2-6x^4 &> 0 \\ 2 &> 6x^4 \\ \frac{1}{3} &> x^4 \\ -\frac{1}{\sqrt[4]{3}} < x &< \frac{1}{\sqrt[4]{3}} \end{aligned}$$

f is concave up on $\left(-\frac{1}{\sqrt[4]{3}}, \frac{1}{\sqrt[4]{3}}\right)$.

(i) (5 points) On what open intervals is f concave down?

Solution: $(1+x^4)^2 > 0$ for all x so $f''(x)$ is positive if and only if $2-6x^4 < 0$.

$$\begin{aligned} 2-6x^4 &< 0 \\ 2 &< 6x^4 \\ \frac{1}{3} &< x^4 \\ x &< -\frac{1}{\sqrt[4]{3}} \text{ or } \frac{1}{\sqrt[4]{3}} < x \end{aligned}$$

f is concave down on $\left(-\infty, -\frac{1}{\sqrt[4]{3}}\right)$ and $\left(\frac{1}{\sqrt[4]{3}}, \infty\right)$.

(j) (5 points) Does f have any horizontal or vertical asymptoptes? If so what are their equations?

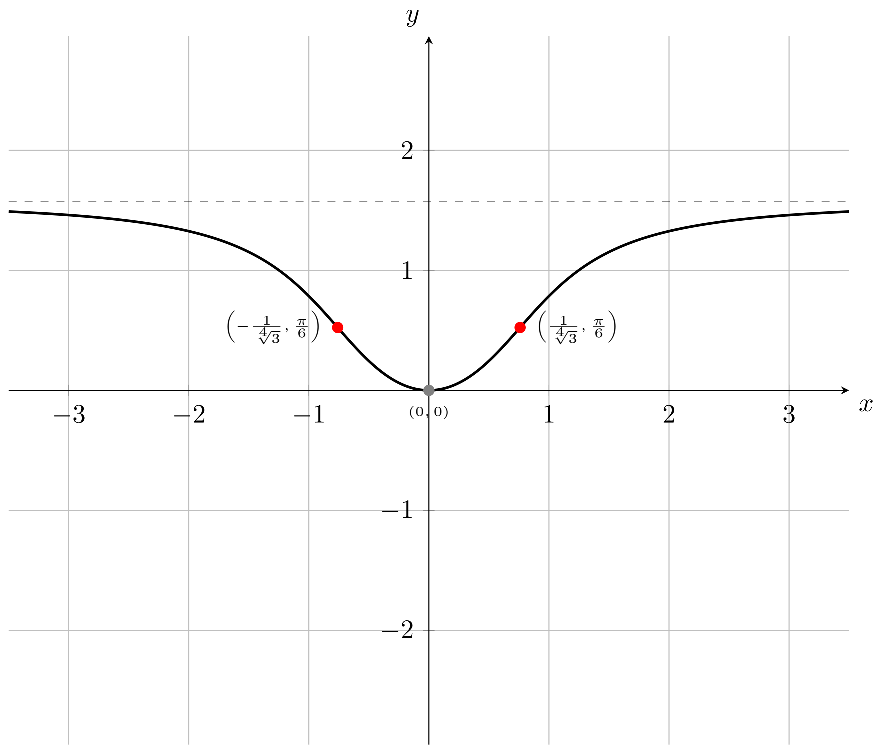
Solution: f is continuous on $(-\infty, \infty)$ so it has no vertical asymptotes.

$$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \arctan(x^2) \\ &= \tfrac{\pi}{2} \\ \lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} \arctan(x^2) \\ &= \tfrac{\pi}{2} \end{aligned}$$

f has no vertical asymptotes and a single horizontal asymptote with equation $y = \frac{\pi}{2}$.

(k) (5 points) Graph f

Solution:



7. Evaluate the following limits using any technique you like.

(a) (5 points) $\lim_{x \to 0} \dfrac{3x^2}{\sin x}$

Solution:

$$\begin{aligned} \lim_{x \to 0} \frac{3x^2}{\sin x} &= \lim_{x \to 0} \frac{6x}{\cos x} \\ &= \lim_{x \to 0} \frac{0}{1} \\ &= \boxed{0} \end{aligned}$$

(b) (5 points) $\lim_{x \to \infty} \sqrt[x]{x}$

Solution:

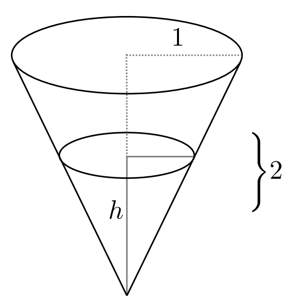
$$\begin{aligned}
\lim_{x\to\infty} \sqrt[x]{x} &= \lim_{x\to\infty} x^{\frac{1}{x}} \\
&= \lim_{x\to\infty} e^{\frac{\ln x}{x}} \\
&= e^{\lim_{x\to\infty} \frac{\ln x}{x}} \\
&= e^{\lim_{x\to\infty} \frac{x^{-1}}{1}} \\
&= e^{\frac{0}{1}} \\
&= \boxed{1}
\end{aligned}$$

(c) (5 points) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x$

Solution:

$$\begin{aligned}
\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x &= \lim_{x\to\infty} e^{x \ln\left(1+\frac{2}{x}\right)} \\
&= e^{\lim_{x\to\infty} x \ln\left(1+\frac{2}{x}\right)} \\
&= e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{2}{x}\right)}{x^{-1}}} \\
&= e^{\lim_{x\to\infty} \frac{\left(1+\frac{2}{x}\right)^{-1}(-2)x^{-2}}{(-1)x^{-2}}} \\
&= e^{\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^{-1}(2)} \\
&= e^{(1^{-1}\cdot 2)} \\
&= \boxed{e^2}
\end{aligned}$$

8. Water is pouring into a cone with a height of 2 meters and a radius of 1 meter at a rate of $3\ \mathrm{m}^3/\mathrm{s}$.



(a) (5 points) Give an equation relating the height h of the water to the volume V of water in the cone. *(Hint: The volume of a cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.)*

Solution: By similar triangles,

$$\frac{h}{r} = \frac{2}{1}$$

so

$$r = \frac{h}{2}$$

Therefore,

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$\boxed{V = \frac{\pi}{12}h^3}$$

(b) (5 points) When the height of the water is 1 meter how fast is the water level rising.

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(V = \frac{\pi}{12}h^3\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{12} \cdot 3h^2 \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t}\frac{4}{\pi h^2}$$

$\frac{\mathrm{d}V}{\mathrm{d}t} = 3$ and $h = 1$ so

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 3 \cdot \frac{4}{\pi \cdot 1^2}$$

$$= \boxed{\tfrac{12}{\pi}\,\mathrm{m/s}}$$

9. The height h and radius r of a circular cylinder are both greater than or equal to 0 and have a sum of 1 meter.

(a) (5 points) Express the volume V of the cylinder as a function of its radius r. Include the **domain** for the radius. *(Hint: The volume of a cylinder with height h and radius r is $\pi r^2 h$.)*

Solution:

$$h + r = 1$$

$$h = 1 - r$$

$$V = \pi r^2 h$$

so

$$V = \pi r^2(1 - r)$$

For the domain, $h = 1 - r \geq 0$ so $1 \geq r$. Also $r \geq 0$ so $0 \leq r \leq 1$.

$$\boxed{V = \pi r^2 - \pi r^3 \quad r \in [0, 1]}$$

(b) (5 points) Is there a radius in the domain above which gives the maximum volume for the cylinder? Why or why not? In particular do any theorems from class or the book apply to this situation?

Solution: Volume is a continuous function of the radius on the closed interval $[0, 1]$ so by the Extreme Value Theorem, there is a radius in the domain $[0, 1]$ which gives the maximum volume for the cylinder.

(c) (5 points) Find the radius r of the cylinder with the maximum volume.

Solution:

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 2\pi r - 3\pi r^2 = \pi r(2 - 3r)$$

Critical points satisfy $0 = \pi r(2 - 3r)$. So the critical points are 0 and $\frac{2}{3}$.
Endpoints are 0 and 1.

$$V(0) = \pi 0^2 - \pi 0^3 = 0$$
$$V\left(\tfrac{2}{3}\right) = \pi\left(\tfrac{2}{3}\right)^2 - \pi\left(\tfrac{2}{3}\right)^3 = \tfrac{4\pi}{27}$$
$$V(1) = \pi 1^2 - \pi 1^3 = 0$$

The maximum volume occurs when the radius is $\boxed{\frac{2}{3}\,\mathrm{m}}$.

(d) (5 points) What is the maximum volume?

Solution: The maximum volume is $\boxed{\frac{4\pi}{27}\,\mathrm{m}^3}$.