HOMEWORK #10 SOLUTIONS

4.7.6) Since $\frac{\sin x}{x} \to 1$ as x approaches 0, we know that $\sin x$ and x approach zero at the same rate. So $\sin^2 x$ will go to 0 faster than x, and we expect $\frac{\sin^2 x}{x} \to 0$.

Since $\lim_{x\to 0} \sin^2 x = 0$ and $\lim_{x\to 0} x = 0$, we have $\lim_{x\to 0} \frac{\sin^2 x}{x}$ is of the form $\frac{0}{0}$. So we can use l'Hopital's Rule to evaluate this limit.

And so
$$\lim_{x\to 0} \frac{\sin^2 x}{x} \overset{H}{=} \lim_{x\to 0} \frac{2\sin x\cos x}{1}$$
$$= 2 \cdot \lim_{x\to 0} \sin x \cdot \lim_{x\to 0} \cos x$$
$$= 2 \cdot 0 \cdot 1 = 0.$$

4.7.7) Since $x^{1/3} \to 0$ more slowly than x, it goes more slowly than $\sin x$ (which approaches zero at the same rate as x)

Thus we would expect $\frac{\sin x}{x^{1/3}} \to 0$, just as in the previous problem.

Using l'Hopital, we see that
$$\lim_{x\to 0} \frac{\sin x}{x^{1/3}} \overset{H}{=} \lim_{x\to 0} \frac{\cos x}{\frac{1}{3}x^{-2/3}}$$
$$= \lim_{x\to 0} 3x^{2/3} \cos x$$
$$= 3 \cdot 0 \cdot 1 = 0.$$

4.7.22) $\lim_{x\to1} \ln x = 0$, and $\lim_{x\to1} (x^2-1) = 0$, so

$$\lim_{x\to1} \frac{\ln x}{x^2-1} \overset{H}{=} \lim_{x\to1} \frac{1/x}{2x}$$

$$= \lim_{x\to1} \frac{1}{2x^2} = \frac{1}{2} .$$

4.7.24) $\lim_{n\to\infty} \sqrt[n]{n} = \lim_{n\to\infty} n^{1/n}$, which is of the form ∞^0. So we use the "exponential-log" trick.

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln(n^{1/n})}$$

$$= e^{\lim_{n\to\infty} \ln(n)^{1/n}}$$

$$= e^{\lim_{n\to\infty} \frac{1}{n}\cdot\ln(n)}$$

$$= e^{\lim_{n\to\infty} \frac{\ln(n) \nearrow \infty}{n \searrow \infty}}$$

$$\overset{H}{=} e^{\lim_{n\to\infty} \frac{1/n}{1}}$$

$$= e^{\lim_{n\to\infty} \frac{1}{n}} = e^0 = 1 .$$

4.7.36) Since $\lim_{x\to 1} \sin(2x) = \sin(2) \neq 0$ or $\pm\infty$

$\lim_{x\to 1} x = 1 \neq 0$ or $\pm\infty$,

$\lim_{x\to 1} \frac{\sin(2x)}{x}$ is not of "l'Hopital form".

Using the fact that $\frac{\sin(2x)}{x}$ is continuous at 1,

$$\lim_{x\to 1} \frac{\sin(2x)}{x} = \frac{\sin 2}{1} = \sin 2 .$$

4.7.45) We first note that since $\frac{3}{5} < 1$,

$$\lim_{t\to -\infty} \left(\frac{3}{5}\right)^t = \infty .$$

Then $\lim_{t\to -\infty} \left(\frac{3^t+5^t}{2}\right)^{1/t} \overset{①}{=} e^{\lim_{t\to -\infty} \frac{1}{t} \ln\left(\frac{3^t+5^t}{2}\right)} =$

$$= e^{\lim_{t\to -\infty} \frac{\ln(3^t+5^t) - \ln 2}{t}} \quad \begin{matrix} \nearrow -\infty \\ \to -\infty \end{matrix}$$

② $$\overset{H}{=} e^{\lim_{t\to -\infty} \frac{3^t \ln 3 + 5^t \ln 5}{3^t + 5^t}}$$

③ $$= e^{\lim_{t\to -\infty} \frac{3^t \ln 3 + (5^t \ln 3 - 5^t \ln 3) + 5^t \ln 5}{3^t + 5^t}}$$

$$④ = e^{\lim\limits_{t \to -\infty} \frac{(3^t+5^t)\ln 3}{3^t+5^t} + \frac{5^t(\ln 5 - \ln 3)}{3^t+5^t}}$$

$$⑤ = e^{\lim\limits_{t \to -\infty} \ln 3 + \lim\limits_{t \to -\infty} \frac{\ln 5 - \ln 3}{\left(\frac{3}{5}\right)^t + 1}}$$

$$= e^{\ln 3 + \frac{\ln 5 - \ln 3}{\lim\limits_{t \to \infty} (3/5)^t + 1}}$$

$$= e^{\ln 3 + 0}$$

$$= 3.$$

Notes :

① Since this is of the form 0^0, we use the "exponential - log" trick.

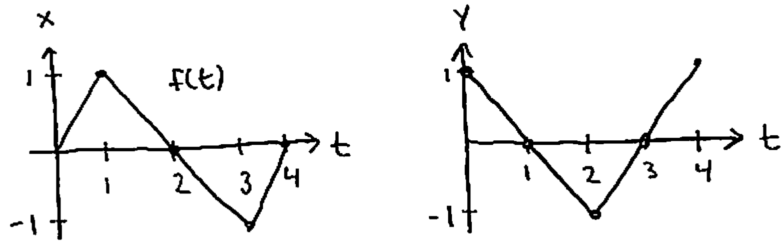
② Work out the derivatives and simplify.

③ Useful trick of adding 0 to make things "nice"

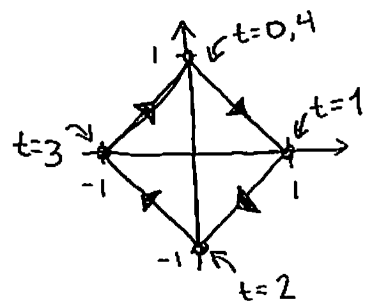
④ 2 or 3 steps of algebra

⑤ Multiply top and bottom of the second term by 5^{-t} and simplify.

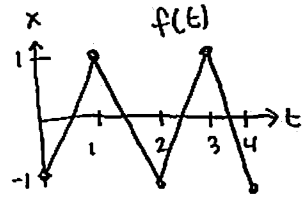
4.8.1) For $x = f(t)$ and $y = g(t)$ defined by the graphs

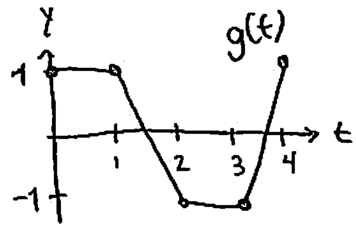


We have the position of the particle in the x,y- plane given by the following curve.



4.8.4)





For $P(t) = (f(t), g(t))$, we have

$P(0) = (-1, 1)$

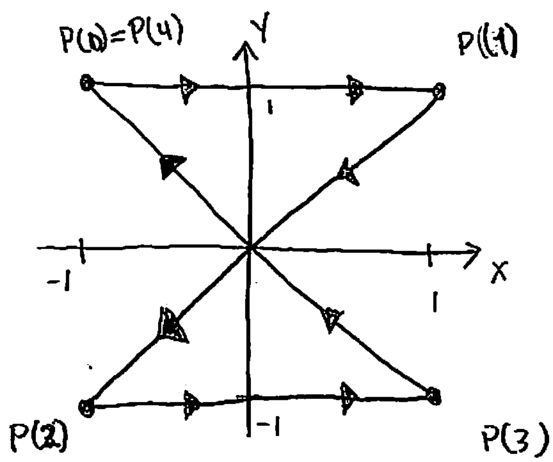
$P(1) = (1, 1)$

$P(2) = (-1, -1)$

$P(3) = (1, -1)$

$P(4) = (-1, 1)$.

Connect the dots,



4.8.22) Using Ex 11 as a guide, we'd like to say that

$$\begin{aligned} x &= 3\cos t \\ y &= 7\sin t \end{aligned}$$, would work.

However we must start at $(-3,0)$, and so

$$\begin{aligned} x &= -3\cos t \\ y &= 7\sin t \end{aligned}$$ would make that happen.

In order to move counter clockwise, $\left.\frac{dy}{dt}\right|_{t=0} < 0$,

And $\frac{dy}{dt} = 7\cos t$,

$\left.\frac{dy}{dt}\right|_{t=0} = 7$ doesn't give us that.

However $y = -7\sin t \Rightarrow \frac{dy}{dt} = -7\cos t$ will give us the desired negative derivative at $t=0$.

So $$\begin{aligned} x &= -3\cos t \\ y &= -7\sin t \end{aligned}$$ is the correct parameterization.

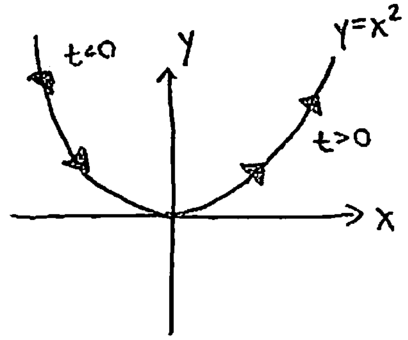
4.8.38) For (a) $x = t$, $y = t^2$

(b) $x = t^2$, $y = t^4$

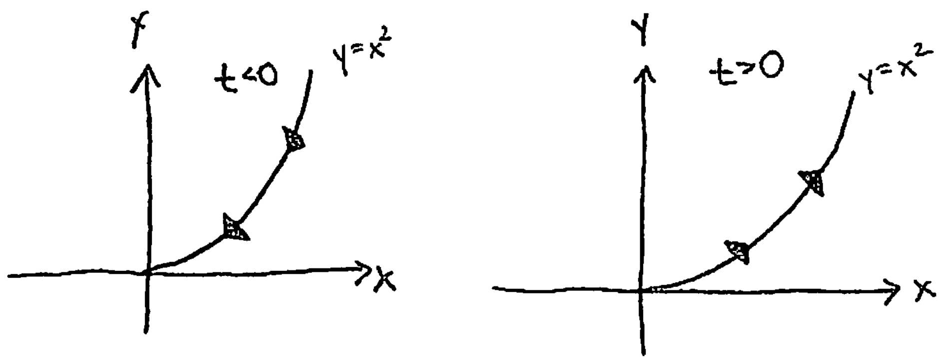
(c) $x = t^3$, $y = t^6$

we are tempted to think, because $y = x^2$ in each case, that the curves would be the same.

But (a)



(b)



In addition, even for $t > 0$, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are different in case (a) and (b): For "small" values of t, the particle is moving more slowly in case (b), and large values of t give a faster particle in case (b).

Case (c) is comparable to case (a), but again, the particle is moving at a different speed in the two cases.

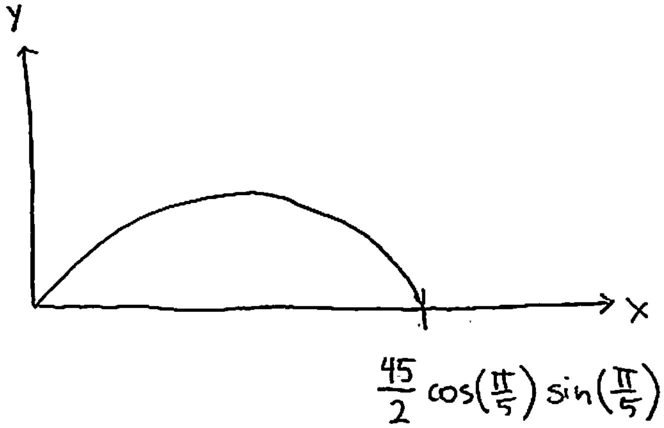
4.8.48) (a) $x = 60\cos\left(\frac{\pi}{5}\right)\cdot t$

$y = 60\sin\left(\frac{\pi}{5}\right)\cdot t - 16t^2$

(I like radians. Degree notation is for LA majors.)

(b) $t = \frac{1}{60\cos\left(\frac{\pi}{5}\right)}x$ gives us

$$y = \tan\left(\frac{\pi}{5}\right)x - \frac{16}{360\cos^2(\pi/5)}x^2$$



(c) Solve for $60\sin\left(\frac{\pi}{5}\right)t - 16t^2 = 0$, ruling out $t=0$ (since that is when the ball was kicked), to get

$$t_f = \frac{15}{4}\sin\left(\frac{\pi}{5}\right) \approx 2.204 \text{ sec}$$

(d) Set $\tan\left(\frac{\pi}{5}\right)x - \frac{16}{360\cos(\pi/5)}x^2 = 0$ and solve to

find $x_f = \frac{45}{2}\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{5}\right) \approx 10.699$ ft

e) For $|v|$ = speed,

$$|v|(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$= \sqrt{360\cos^2\left(\frac{\pi}{5}\right) + \left(60\sin\left(\frac{\pi}{5}\right) - 32t\right)^2}$$

So $|v|(1) = \sqrt{360\cos^2\left(\frac{\pi}{5}\right) + \left(60\sin\left(\frac{\pi}{5}\right) - 32\right)^2}$

≈ 15.694 ft/sec

50) For $x = t^3 + t$, $y = t^2$,

$\frac{dx}{dt} = 3t^2 + 1$, $\frac{dy}{dt} = 2t$.

So $\frac{dy}{dx} = \frac{2t}{3t^2+1}$.

then $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{2(3t^2+1) - 2t(6t)}{(3t^2+1)^2}}{3t^2+1}$$

$$= \frac{2(1-3t^2)}{(3t^2+1)^3}$$

At $t=1$, $\frac{d^2y}{dx^2} = \frac{-4}{4^3} = -\frac{1}{16} < 0$.

So the curve is concave down.