Homework 6 solutions

3.1.41) $f(x) = \dfrac{ax+b}{x} = \dfrac{ax}{x} + \dfrac{b}{x} = a + bx^{-1}$ (algebra)

$$\frac{df}{dx} = \frac{d}{dx}\left[a + bx^{-1}\right] = \frac{d}{dx}[a] + \frac{d}{dx}\left[bx^{-1}\right]$$ (sum rule)

$$= \frac{d}{dx}[a] + b\frac{d}{dx}\left[x^{-1}\right]$$ (constant multiple rule)

$$= 0 + b \cdot (-1)x^{-1-1}$$ (derivative of a constant, power rule)

$$= -b\,x^{-2} = -\frac{b}{x^2}$$ (algebra)

3.1.43) $g(t) = \dfrac{\sqrt{t}\,(1+t)}{t^2} = \dfrac{t^{1/2}(1+t)}{t^2} = \dfrac{t^{1/2} + t^{3/2}}{t^2} =$

$$= \frac{t^{1/2}}{t^2} + \frac{t^{3/2}}{t^2} = t^{-3/2} + t^{-1/2}$$ (algebra)

$$g'(t) = \left(t^{-3/2} + t^{-1/2}\right)' = \left(t^{-3/2}\right)' + \left(t^{-1/2}\right)'$$ (sum rule)

$$= \left(-\frac{3}{2}\right)t^{-3/2-1} + \left(-\frac{1}{2}\right)t^{-1/2-1}$$ (power rule)

$$= -\frac{3}{2}t^{-5/2} - \frac{1}{2}t^{-3/2}$$ (algebra)

3.1.58) On what intervals is the function $f(x) = x^4 - 4x^3$ both decreasing and concave up?

Solution: f is decreasing when $f' < 0$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$f'(x) = 0$ at $x = 0$ and $x = 3$

		0		3	
$4x^2$	+		+		+
$x-3$	−		−		+
$f'(x)$	−		−		+

$f'(x) < 0$ on $(-\infty, 0) \cup (0, 3) = A$

f is concave up when $f'' > 0$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

$f''(x) = 0$ at $x = 0$ and $x = 2$

		0		2	
$12x$	−		+		+
$(x-2)$	−		−		+
$f''(x)$	+		−		+

$f''(x) > 0$ on $(-\infty, 0) \cup (2, \infty) = B$

f is both decreasing and concave up on $A \cap B = (-\infty, 0) \cup (2, 3)$

3.3.26) $f(z) = \dfrac{3z^2}{5z^2+7z} = \dfrac{z \cdot 3z}{z(5z+7)} = \dfrac{3z}{5z+7}$ (algebra)

$f'(z) = \left(\dfrac{3z}{5z+7}\right)' = \dfrac{(3z)' \cdot (5z+7) - (3z) \cdot (5z+7)'}{(5z+7)^2}$ (quotient rule)

$= \dfrac{3 \cdot (5z+7) - 3z \cdot 5}{(5z+7)^2}$ (power rule, constant multiple rule, derivative of constant)

$= \dfrac{15z + 21 - 15z}{(5z+7)^2}$

$= \dfrac{21}{(5z+7)^2}$ (algebra)

3.3.40) $f(t) = e^{-t} = \dfrac{1}{e^t}$ (algebra)

$\dfrac{df}{dt} = \dfrac{d}{dt}\left[\dfrac{1}{e^t}\right] = \dfrac{\frac{d}{dt}[1] \cdot e^t - 1 \cdot \frac{d}{dt}[e^t]}{(e^t)^2}$ (quotient rule)

$= \dfrac{0 \cdot e^t - 1 \cdot e^t}{(e^t)^2}$ (derivative of constant, derivative of e^t)

$= \dfrac{-e^t}{(e^t)^2} = -\dfrac{1}{e^t} = -e^{-t}$ (algebra)

3.3.48) Suppose f and g are differentiable functions with the values

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	3	4	5	-2

For each of the following functions h, find $h'(2)$

(a) $h(x) = f(x) + g(x)$

$h'(x) = [f(x) + g(x)]' = f'(x) + g'(x)$ (sum rule)

$h'(2) = f'(2) + g'(2) = 5 + (-2) = 3$ (evaluation of $h'(x)$ at $x=2$)

(b) $h(x) = f(x) \cdot g(x)$

$h'(x) = [f(x) \cdot g(x)]' = f'(x)g(x) + f(x)g'(x)$ (product rule)

$h'(2) = f'(2) \cdot g(2) + f(2) \cdot g'(2)$

$= 5 \cdot 4 + 3 \cdot (-2)$

$= 20 - 6 = 14$

(c) $h(x) = \dfrac{f(x)}{g(x)}$

$$h'(x) = \left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$ (quotient rule)

$$h'(2) = \frac{f'(2)g(2) - f(2)g'(2)}{(g(2))^2} = \frac{5 \cdot 4 - 3(-2)}{4^2} = \frac{20+6}{16} = \frac{26}{16} = \frac{13}{8}$$

3.3.50) Find a possible formula for a function $y = f(x)$ such that

$$y' = f'(x) = 10x^9 e^x + x^{10} e^x$$

Solution: Noticing $10x^9 = \frac{d}{dx}[x^{10}]$, and

$$e^x = \frac{d}{dx}[e^x],$$

we see
$$f'(x) = \frac{d}{dx}[x^{10}] \cdot e^x + x^{10} \frac{d}{dx}[e^x]$$

$$= \frac{d}{dx}[x^{10} \cdot e^x]$$

So for $y = x^{10} e^x$, we get

$y' = 10x^9 e^x + x^{10} e^x$, as desired.

In fact for any constant c, the function

$y = x^{10} e^x + C$ will have the desired derivative.

3.4.25) $z(x) = \sqrt[3]{2^x+5} = (2^x+5)^{1/3}$.

For $u(x) = 2^x + 5$, we have

$z(x) = [u(x)]^{1/3}$, so

$$\frac{dz}{dx} = \frac{d}{dx}\left[[u(x)]^{1/3}\right]$$

$$= \frac{1}{3}[u(x)]^{-2/3} \cdot \frac{d}{dx}[u(x)]$$

$$= \frac{1}{3}(2^x+5)^{-2/3} \cdot \frac{d}{dx}[2^x+5]$$

$$= \frac{1}{3}(2^x+5)^{-2/3} \cdot \left(\frac{d}{dx}[2^x] + \frac{d}{dx}[5]\right)$$

$$= \frac{1}{3}(2^x+5)^{-2/3} \cdot (2^x \ln 2 + 0)$$

$$= \frac{1}{3}(2^x+5)^{-2/3} \cdot 2^x \ln 2$$

3.4.32) $h(x) = \sqrt{\dfrac{x^2+9}{x+3}}$; Let $f(x) = \sqrt{x}$, $g(x) = \dfrac{x^2+9}{x+3}$.

Then $f'(x) = \frac{1}{2} x^{-1/2} = \dfrac{1}{2\sqrt{x}}$ by the power rule, and

$$g'(x) = \frac{2x(x+3) - (x^2+9)\cdot 1}{(x+3)^2}$$

$$= \frac{2x^2 + 6x - x^2 - 9}{(x+3)^2}$$

$$= \frac{x^2+6x-9}{(x+3)^2}$$, by the quotient rule.

Since $h(x) = f(g(x))$, we have

$h'(x) = f'(g(x)) \cdot g'(x)$ by the chain rule

$$= f'\left(\frac{x^2+9}{x+3}\right) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{\frac{x^2+9}{x+3}}} \cdot \frac{x^2+6x-9}{(x+3)^2}$$

$$= \frac{1}{2}\sqrt{\frac{x+3}{x^2+9}} \cdot \frac{x^2+6x-9}{(x+3)^2}$$

algebra / "plugging things in"

3.4.66) Find the mean and variance of the normal distribution of statistics using parts (a) and (b) with $m(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$

(a) Mean $= m'(0)$

(b) Variance $= m''(0) - (m'(0))^2$

Translation: For $m(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$, find

(a) $m'(0)$, and

(b) $m''(0) - (m'(0))^2$.

Solution: Let $u = \mu t + \frac{\sigma^2}{2} t^2$. Then

$u' = \frac{du}{dt} = \mu + \sigma^2 t$ by the power rule (μ and σ are constants)

$u'' = \frac{d^2u}{dt^2} = 0 + \sigma^2 = \sigma^2$

And we have $u(0) = 0$, $u'(0) = \mu$, $u''(0) = \sigma^2$

Then $m(t) = e^u$,

$m'(t) = \frac{d}{dt}[e^u] = e^u \cdot \frac{du}{dt} = e^u \cdot u'$, so

$m'(0) = e^{u(0)} \cdot u'(0) = e^0 \cdot \mu = 1 \cdot \mu = \mu$.

And $m''(t) = \frac{d}{dt}[e^u \cdot u'] = \frac{d}{dt}[e^u] \cdot u' + e^u \cdot \frac{d}{dt}[u']$

$= (e^u \cdot u') \cdot u' + e^u \cdot u''$, so

$m''(0) - (m'(0))^2 = \left(e^{u(0)} \cdot u'(0) \cdot u'(0) + e^{u(0)} \cdot u''(0)\right) - \mu^2$

$= (e^0 \cdot \mu^2 + e^0 \cdot \sigma^2) - \mu^2 = \sigma^2$

3.4.80) For $m = \dfrac{m_o}{\sqrt{1 - v^2/c^2}}$,

(a) Find $\dfrac{dm}{dv}$.

Solution: $m = \dfrac{m_o}{\sqrt{1 - v^2/c^2}} = \sqrt{\dfrac{m_o^2}{\frac{1}{c^2}(c^2 - v^2)}} = \sqrt{\dfrac{m_o^2 c^2}{c^2 - v^2}} =$

$= \dfrac{m_o c}{\sqrt{c^2 - v^2}} = m_o c\,(c^2 - v^2)^{-1/2}$. (algebra)

So $\dfrac{dm}{dv} = m_o c \cdot \dfrac{d}{dv}\left[(c^2 - v^2)^{-1/2}\right]$

$= m_o c \cdot \left(-\frac{1}{2}\right)(c^2 - v^2)^{-3/2} \cdot (-2v)$ (power, chain rules)

$= m_o c v \cdot (c^2 - v^2)^{-3/2}$

(b) In terms of physics, what does $\dfrac{dm}{dv}$ tell you?

A: $\dfrac{dm}{dv}$ tells you how much the mass is increasing at a particular velocity.

3.4.86) Find and simplify $\frac{d^2}{dx^2}\left(\frac{f(x)}{g(x)}\right)$.

Solution: $$\frac{d^2}{dx^2}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left[\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\right]$$

$$= \frac{d}{dx}\left[\frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}\right] \qquad \text{(quotient rule)}$$

$$= \frac{\frac{d}{dx}[f'(x)g(x) - f(x)g'(x)]\cdot[g(x)]^2 - (f'(x)g(x) - f(x)g'(x))\frac{d}{dx}[(g(x))^2]}{[g(x)]^4}$$

$$= \frac{\left(\frac{d}{dx}[f'(x)g(x)] - \frac{d}{dx}[f(x)g'(x)]\right)\cdot[g(x)]^2 - [f'(x)g(x) - f(x)g'(x)]\cdot 2g(x)\cdot g'(x)}{[g(x)]^4}$$

$$= \frac{\left([f''(x)g(x) + f'(x)g'(x)] - [f'(x)g'(x) + f(x)g''(x)]\right)[g(x)]^2 - [2[g(x)]^2 f'(x)g'(x) - 2f(x)g(x)[g'(x)]^2]}{[g(x)]^4}$$

$$= \frac{[f''(x)g(x) - f(x)g''(x)][g(x)]^2 - [2g(x)f'(x)g'(x) - 2f(x)[g'(x)]^2]\,g(x)}{[g(x)]^4}$$

$$= \frac{f''(x)[g(x)]^2 - f(x)g(x)g''(x) - 2f'(x)g(x)g'(x) + 2f(x)[g'(x)]^2}{[g(x)]^3}.$$

3.5.19) $f(x) = \tan(\sin x)$

$= \tan u$, where $u = \sin x$ so $\frac{du}{dx} = \cos x$.

$f'(x) = \sec^2 u \cdot \frac{du}{dx}$

$= \sec^2(\sin x) \cdot \cos x$.

3.5.31) $k(\alpha) = \sin^5\alpha \cos^3\alpha$

$= u^5 v^3$, where $u = \sin\alpha$, $v = \cos\alpha$

$u' = \cos\alpha$ $v' = -\sin\alpha$.

$k'(\alpha) = (5u^4 \cdot u')v^3 + u^5 \cdot (3v^2 \cdot v')$, from the product, power, and chain rules

$= 5\sin^4\alpha \cdot \cos\alpha \cdot \cos^3\alpha + 3\sin^5\alpha \cdot \cos^2\alpha \cdot (-\sin\alpha)$

$= \sin^4\alpha \cos^2\alpha \left(5\cos^2\alpha - 3\sin^2\alpha\right)$

3.5.42) Find a possible formula for the function $g(x)$ such that

$$g'(x) = \frac{e^x \sin x - e^x \cos x}{(\sin x)^2}.$$

Solut on : Rembering $\frac{d}{dx}[e^x] = e^x$, $\frac{d}{dx}[\sin x] = \cos x$.

If we let $f(x) = e^x$ and $g(x) = \sin x$, then

$f'(x) = e^x$ and $g'(x) = \cos x$, and

$$g'(x) = \frac{f'(x) \cdot g(x) - f(x) g'(x)}{[g(x)]^2},$$ which is the exact

formula for the quotient-rule derivative of $\frac{f(x)}{g(x)}$.

So for $g(x) = \frac{e^x}{\sin x}$, we get the desired derivative.

We will, in fact, get the desired $g'(x)$ when

$g(x) = \frac{e^x}{\sin x} + C$ for any constant C.

3.5.53 a) Using the definition of the derivative and the identity

$$\sin(a+b) = \sin a \cos b + \cos a \sin b,$$

we see that, for $f(x) = \sin x$;

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \cdot \sin h}{h}$$

(assuming both of these limits exist!)

$$= \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

(since x is fixed, sinx, cosx are constant with respect to the limit)