1. (7 points) For each of the following, decide whether the statement is True or False. Circle T for *True* or F for *False*. You do not need to show any work.

   T F The graph of $f(x-2)$ is the graph of $f(x)$ shifted to the left by 2 units.

   T F If $g(x) = \dfrac{x^2-2}{x^2-4}$, then $g(x)$ has a vertical asymptote at $x = 2$.

   T F The amplitude of $\sin(3\pi x)$ is 3.

   T F A 7th degree polynomial can have as many as 8 zeros (roots).

   T F The Second Derivative test always tells us if a critical point is a local maximum or a local minimum.

   T F The derivative of the position function is acceleration.

   T F A graph with a vertical tangent line at $x = a$ cannot be the graph of a function that is differentiable at $x = a$.

2. (8 points) Use the **DEFINITION OF THE DERIVATIVE** to find the derivative of $f(x) = \dfrac{1}{x+1}$
(Show all working. You will receive no credit if you use 'shortcuts'.)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x+1}{(x+h+1)(x+1)} - \frac{x+h+1}{(x+h+1)(x+1)}}{h}$$

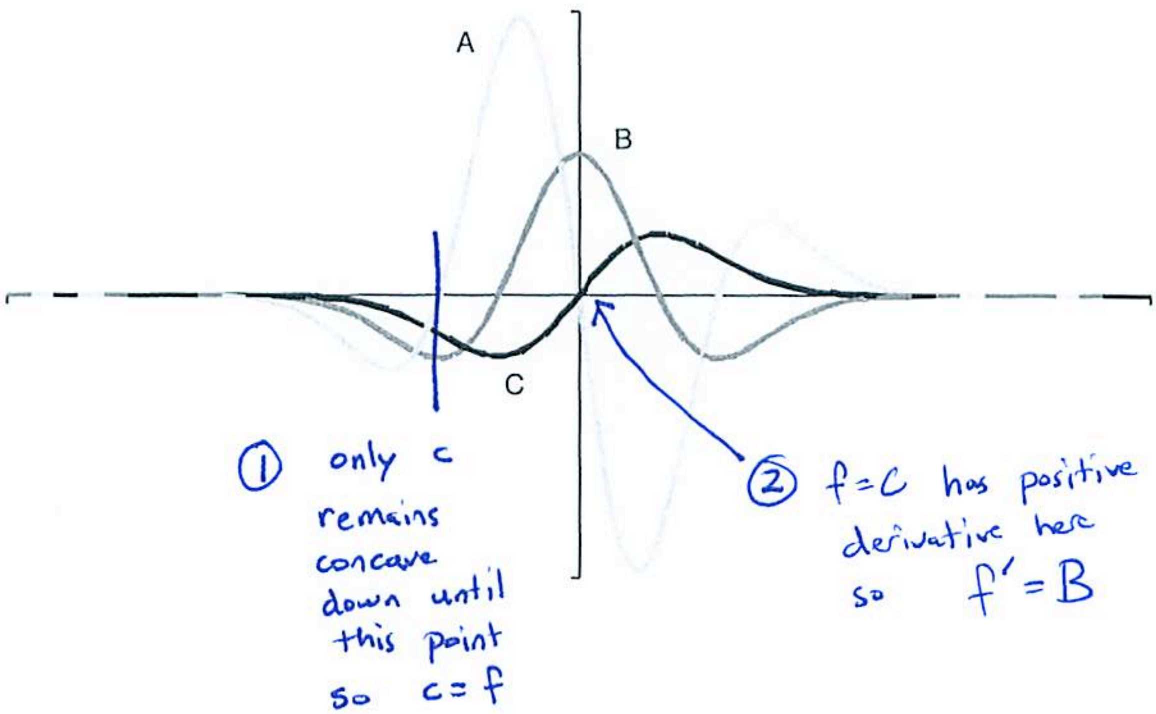
$$= \lim_{h\to 0} \frac{x+1-x-h-1}{(x+h+1)(x+1)h}$$

$$= \lim_{h\to 0} \frac{-h}{(x+h+1)(x+1)h}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h+1)(x+1)}$$

$$= \frac{-1}{(x+1)^2}$$

3. (6 points) On the following axes, the graphs labelled by A, B, and C are graphs of a function f, its derivative f', and its second derivative f'', but not necessarily in that order. You must identify which is which. You need not show your work.



The graph of f is <u>C</u>.

The graph of f' is <u>B</u>.

The graph of f'' is <u>A</u>.

4. (a) (10 points) Use implicit differentiation to find $\frac{dy}{dx}$ when (x, y) lies on the curve $1 + x^2y = 2x\cos(y)$.

$$\frac{d}{dx}\left(1 + x^2 y = 2x \cos y\right)$$

$$0 + 2xy + x^2\frac{dy}{dx} = 2\cos y + 2x(-\sin y)\frac{dy}{dx}$$

$$x^2\frac{dy}{dx} + 2x(\sin y)\frac{dy}{dx} = 2\cos y - 2xy$$

$$\frac{dy}{dx}\left(x^2 + 2x\sin y\right) = 2\cos y - 2xy$$

$$\frac{dy}{dx} = \frac{2\cos y - 2xy}{x^2 + 2x\sin y}$$

(b) Find the equation of the tangent line to the curve at the point $\left(\frac{1}{2}, 0\right)$.

at $\left(\frac{1}{2}, 0\right)$

$$\frac{dy}{dx} = \frac{2\cos 0 - 2\left(\frac{1}{2}\right)\cdot 0}{\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)\sin 0}$$

$$= \frac{2}{\left(\frac{1}{4}\right)} =$$

$$= 8$$

line is $y - 0 = 8\left(x - \frac{1}{2}\right)$

$$y = 8\left(x - \frac{1}{2}\right)$$

5. (12 points) Let $f(x) = \dfrac{x^3}{3} + \dfrac{x^2}{2} - 6x + 5$.

(a) Find the y-intercept of $f(x)$.

$$f(0) = \frac{0^3}{3} + \frac{0^2}{2} - 6(0) + 5 = 5$$

Answer: $y =$ 5

(b) Determine the interval(s) on which $f(x)$ is increasing and on which $f(x)$ is decreasing. $f'(x) = x^2 + x - 6 = (x+3)(x-2)$

		-3		2	
$x+3$	$-$		$+$		$+$
$x-2$	$-$		$-$		$+$
$(x+3)(x-2)$	$+$		$-$		$+$

Answer: increasing $(-\infty, -3]$ and $[2, \infty)$

Answer: decreasing $[-3, 2]$

Note: $(-\infty, -3)$ $(2, \infty)$ $(-3, 2)$ are also acceptable

(c) Find the coordinate pair of each local maximum and local minimum of $f(x)$. by 1^{st} derivative test

$x = -3$ $\quad f(-3) = \dfrac{(-3)^3}{3} + \dfrac{(-3)^2}{2} - 6(-3) + 5 = \dfrac{37}{2}$ loc. max.

$x = 2$ $\quad f(2) = \dfrac{2^3}{3} + \dfrac{2^2}{2} - 6(2) + 5 = -\dfrac{7}{3}$ loc. min.

Answer: local maxima, $(x, y) =$ $\left(-3, \frac{37}{2}\right)$

Answer: local minima, $(x, y) =$ $\left(2, -\frac{7}{3}\right)$

(d) Determine the interval(s) on which $f(x)$ is concave up and on which $f(x)$ is concave down.

$f''(x) = 2x + 1$

		$-\frac{1}{2}$	
$2x+1$	$-$		$+$

$0 = 2x + 1$

$x = -\frac{1}{2}$

Answer: concave up $\left(-\frac{1}{2}, \infty\right)$

Answer: concave down $\left(-\infty, -\frac{1}{2}\right)$

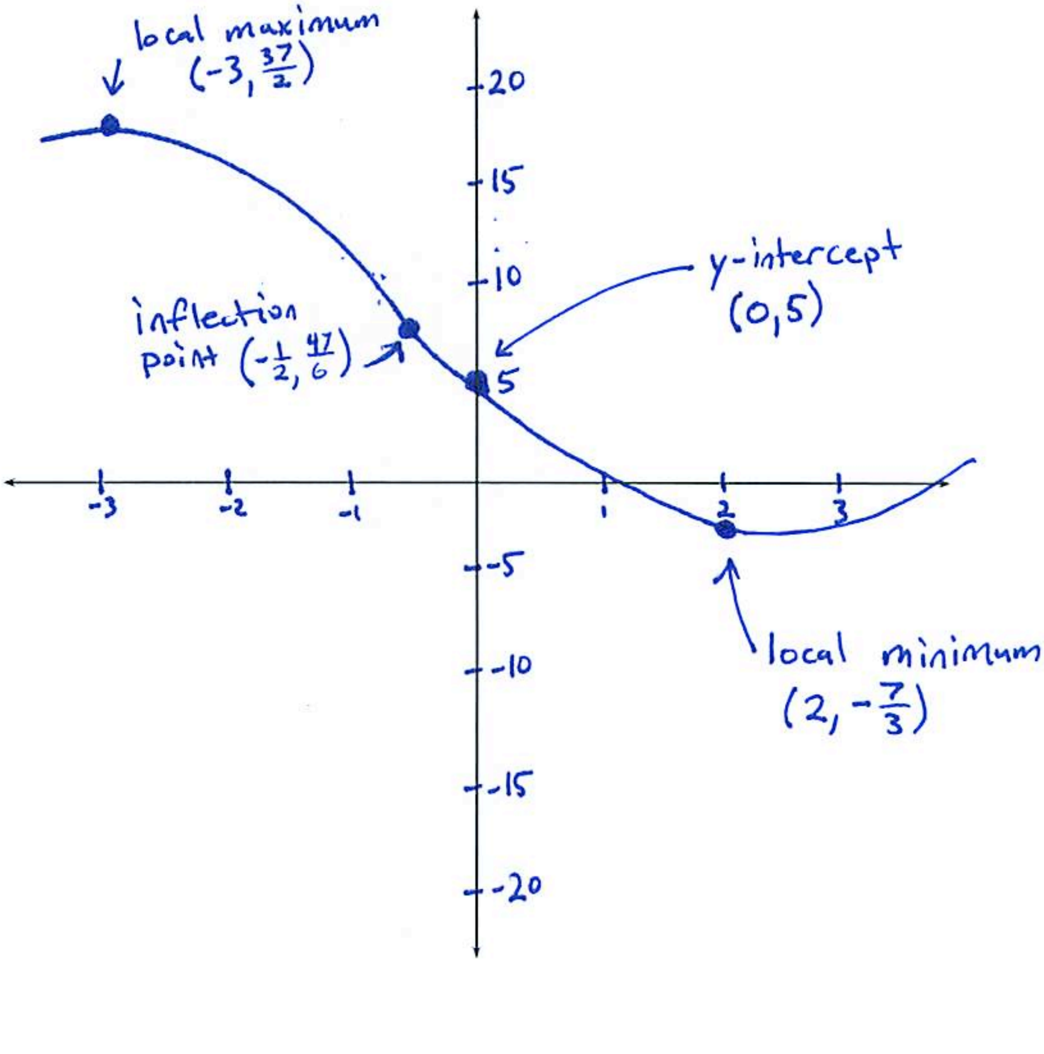
(e) Find the coordinate pair of each inflection point of $f(x)$.

$$x = -\frac{1}{2} \qquad f(x) = \frac{(-\frac{1}{2})^3}{3} + \frac{(-\frac{1}{2})^2}{2} - 6(-\tfrac{1}{2}) + 5$$

$$= \frac{47}{6}$$

Answer: inflection point(s), $(x, y) =$ $\left(-\frac{1}{2}, \frac{47}{6}\right)$

(f) Using (a)-(e), sketch a graph of $f(x)$ on the axes below.
(Be sure to label all local extrema and inflection points.)



6. (10 points) Under ideal conditions the population of a certain bacterium is known to double every three hours. Suppose that there are initially 100 bacteria.

   (a) Find an equation for the population, B, in terms of time t, in hours.

   $$B(t) = 100 \cdot 2^{\left(\frac{t}{3}\right)}$$

   (b) What is the population after 21 hours?

   $$B(21) = 100 \cdot 2^{\frac{21}{3}}$$
   $$= 100 \cdot 2^{7}$$ (This answer is simplified enough)
   $$= 12800$$

   (c) How long will it take the bacteria to reach a population of 900?

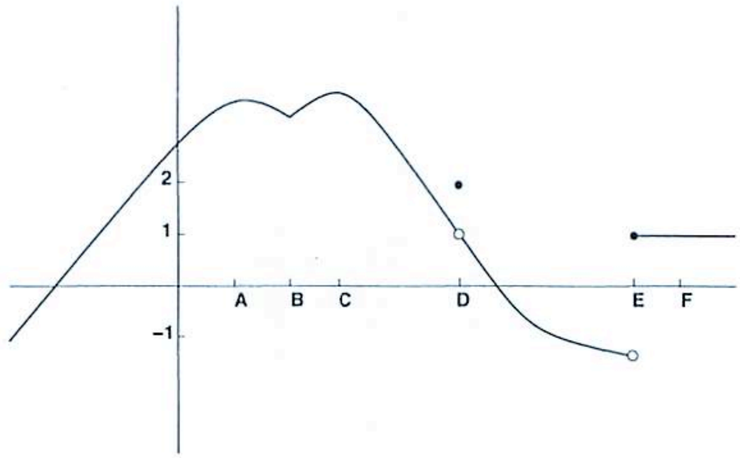
   $$900 = 100 \cdot 2^{\left(\frac{t}{3}\right)}$$
   $$9 = 2^{\left(\frac{t}{3}\right)}$$
   $$\ln 9 = \ln\left(2^{\frac{t}{3}}\right)$$
   $$\ln 9 = \frac{t}{3}\ln 2$$
   $$t = \frac{3\ln 9}{\ln 2}$$

   (d) Compute $B'(10)$. What is the practical meaning of $B'(10)$?

   $$B'(t) = 100\,(\ln 2)\left(\tfrac{1}{3}\right) 2^{\left(\frac{t}{3}\right)}$$
   $$B'(10) = 100\,(\ln 2)\left(\tfrac{1}{3}\right) 2^{\frac{10}{3}}$$

   This means that after 10 hours the population of bacteria is growing at a rate of $100(\ln 2)(\frac{1}{3})2^{\frac{10}{3}}$ bacteria per hour.

7. (9 points) This is a graph of the function $y = f(x)$.



(a) At which values of x is $f(x)$ not continuous?

D, E

(b) At which values of x is $f(x)$ not differentiable?

B, D, E

(c) What is the value of $f'(F)$?

0

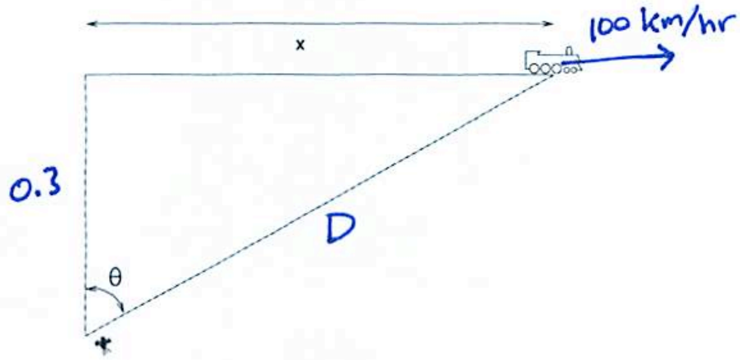
(d) Does $\lim_{x \to D} f(x)$ exist? If so, what is its value?

$\lim_{x \to D} f(x) = 1$

(e) Does $\lim_{x \to E} f(x)$ exist? If so, what is its value?

$\lim_{x \to E} f(x)$ does not exist since the limits from the left and right do not agree.

8. (10 points) A train is moving at 100 km/hr along a straight track. 0.3 km from the track there is a movie camera, focused on the train.



(a) How fast is the distance from the camera to the train changing when the train is 0.5 km from the camera? State the units in your answer.

$$D^2 = x^2 + (0.3)^2$$

$$\frac{d}{dt}\left(D^2 = x^2 + (0.3)^2\right)$$

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dD}{dt} = \frac{x}{D}\cdot\frac{dx}{dt}$$

$$x^2 = D^2 - (.3)^2$$

$$x = \pm\sqrt{D^2 - (.3)^2}$$

(positive root matches picture)

$$\frac{dD}{dt} = \frac{\cancel{\pm}\sqrt{D^2-(.3)^2}}{D}\cdot\frac{dx}{dt}$$

$$\frac{dD}{dt} = \frac{\sqrt{.5^2 - .3^2}}{.5}\cdot 100 \text{ km/hr}$$

(b) How fast is the camera rotating at the moment when the train is 0.5 km from the camera? State the units in your answer.

$$\tan\theta = \frac{x}{.3}$$

$$\frac{d}{dt}\left(\tan\theta = \frac{x}{.3}\right)$$

$$(\sec^2\theta)\frac{d\theta}{dt} = \frac{1}{.3}\frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{.3}\cdot\frac{dx}{dt}$$

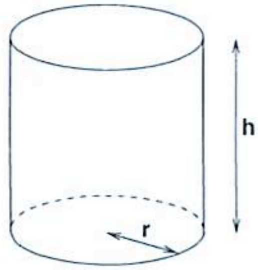
$$\cos\theta = \frac{0.3}{D} = \frac{0.3}{0.5}$$

so

$$\frac{d\theta}{dt} = \frac{\left(\frac{0.3}{0.5}\right)^2}{.3}\cdot 100 \text{ rad/hr}$$

$$= 12 \text{ radians/hr}$$

9. (8 points) Find the minimum amount of sheet metal that can be used to make a cylindrical bucket with a volume of 100 cubic inches. (The bucket has an open top: only the sides and base need to be made of metal. Recall that the volume of a cylinder of radius r and height h is given by the formula $\pi r^2 h$.)



$$A = \underbrace{\pi r^2}_{\text{bottom}} + \underbrace{2\pi r h}_{\text{sides}}$$

$V = \pi r^2 h$ volume is fixed at 100 so $100 = \pi r^2 h$

Therefore $h = \frac{100}{\pi r^2}$

$$A = \pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right)$$

$$= \pi r^2 + \frac{200}{r}$$ on domain $r \in (0,\infty)$

Critical points for A

$$\frac{dA}{dr} = 2\pi r + 200(-1)r^{-2}$$

$r \neq 0$

$$0 = 2\pi r - 200 r^{-2}$$

$$200 r^{-2} = 2\pi r$$

$$200 = 2\pi r^3$$

$$r = \sqrt[3]{\frac{100}{\pi}}$$ critical point

$$\frac{dA}{dr} = \frac{1}{r^2}\left(2\pi r^3 - 200\right)$$

r^{-2}: + +

$2\pi r^3 - 200$: − +

0, $\sqrt[3]{\frac{100}{\pi}}$

$\frac{dA}{dr}$: − +

so A is decreasing on $\left(0, \sqrt[3]{\frac{100}{\pi}}\right)$ and increasing on $\left(\sqrt[3]{\frac{100}{\pi}}, \infty\right)$ so $\sqrt[3]{\frac{100}{\pi}}$ is a global minimum.

$$A\left(\sqrt[3]{\frac{100}{\pi}}\right) = \pi\left(\sqrt[3]{\frac{100}{\pi}}\right)^2 + \frac{200}{\sqrt[3]{\frac{100}{\pi}}}$$ so