Mathematics 152.02
Solutions to Review for Final Exam

1. Suppose that the function $f(t)$ represents the price of a barrel of oil in dollars at time t, measured in days, where $f(0)$ is the price at midnight on December 31, 2005. In words, what is represented by $\frac{1}{90}\int_0^{90} f(t)\,dt$?

   $\frac{1}{90}\int_0^{90} f(t)\,dt$ is the average price of a barrel of oil in dollars during the first 90 days of 2006.

2. Suppose that $f(x)$ is an odd function, $g(x)$ is an even function, and that the following values are known:

$$\int_0^3 f(x)\,dx = 2 \qquad \int_3^5 f(x)\,dx = 1 \qquad \int_0^3 g(x)\,dx = 4 \qquad \int_3^5 g(x)\,dx = 1.$$

   Find $\int_{-3}^5 f(x) + g(x)\,dx$.

$$\begin{aligned}
\int_{-3}^5 f(x) + g(x)\,dx &= \int_{-3}^5 f(x)\,dx + \int_{-3}^5 g(x)\,dx \\
&= \int_{-3}^3 f(x)\,dx + \int_3^5 f(x)\,dx + \int_{-3}^3 g(x)\,dx + \int_3^5 f(x)\,dx \\
&= 0 + \int_3^5 f(x)\,dx + 2\int_0^3 g(x)\,dx + \int_3^5 f(x)\,dx \\
&= 0 + 1 + 2 \cdot 4 + 1 \\
&= \boxed{10}
\end{aligned}$$

3. Find the solution to the initial value problem

$$f'(t) = \frac{t^3+1}{t^3-4t} \qquad f(1) = 2$$

$$\begin{aligned}
f'(t) &= \frac{t^3+1}{t^3-4t} \\
&= 1 + \frac{4t+1}{t^3-4t} \\
&= 1 + \frac{4t+1}{t(t+2)(t-2)} \\
&= 1 + \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2} \\
&= 1 - \frac{1}{4t} - \frac{7}{8(t+2)} + \frac{9}{8(t-2)}
\end{aligned}$$

Thus

$$\begin{aligned}
f(t) &= \int 1 - \frac{1}{4t} - \frac{7}{8(t+2)} + \frac{9}{8(t-2)}\,dx \\
&= t - \frac{\ln|t|}{4} - \frac{7\ln|t+2|}{8} + \frac{9\ln|t-2|}{8} + C
\end{aligned}$$

$f(1) = 2$ implies that

$$\begin{aligned}
2 &= 1 - \frac{\ln|1|}{4} - \frac{7\ln|1+2|}{8} + \frac{9\ln|1-2|}{8} + C \\
C &= 2 - 1 + \frac{\ln|1|}{4} + \frac{7\ln|1+2|}{8} - \frac{9\ln|1-2|}{8} \\
C &= 2 - 1 + \frac{0}{4} + \frac{7\ln 3}{8} - \frac{9 \cdot 0}{8} \\
C &= 1 + \frac{7\ln 3}{8}
\end{aligned}$$

$$\boxed{f(t) = t - \frac{\ln|t|}{4} - \frac{7\ln|t+2|}{8} + \frac{9\ln|t-2|}{8} + 1 + \frac{7\ln 3}{8}}$$

4. Show that the function $f(x) = \frac{1}{x}$ is concave up for $x > 0$. Without using your calculator, show that $\dfrac{2}{3} \leq \displaystyle\int_1^2 \frac{1}{x}\,dx \leq \frac{3}{4}$. (Hint: compute MID(1) and TRAP(1).) Find the SIMP(1) estimate for this integral.

$$f'(x) = -x^{-2}$$
$$f''(x) = 2x^{-3}$$

$x > 0$ implies that $x^{-3} > 0$ which imples that $2x^{-3} > 0$. Thus if $x > 0$ we have $f''(x) > 0$. Therefore f is concave up on the domain $(0, \infty)$.

f is concave up on the interval $[1, 2]$ so $\text{MID}(1) \leq \int_1^2 \frac{1}{x}\,dx \leq \text{TRAP}(1)$

$$\text{MID}(1) = \left(\frac{3}{2}\right)^{-1} \cdot 1 = \frac{2}{3}$$
$$\text{TRAP}(1) = \frac{1^{-1} + 2^{-1}}{2} \cdot 1 = \frac{3}{4}$$

so $\dfrac{2}{3} \leq \displaystyle\int_1^2 \frac{1}{x}\,dx \leq \frac{3}{4}$.

$$\text{SIMP}(1) = \frac{2 \cdot \text{MID}(1) + \text{TRAP}(1)}{3} = \frac{2 \cdot \frac{1}{2} + \frac{3}{4}}{3} = \boxed{\tfrac{7}{12}}$$

5. The function $f(t)$ is known to be decreasing and concave down for $0 \leq t \leq 2$. Put the following estimates for $\displaystyle\int_0^2 f(t)\,dt$ in increasing order: LEFT(10), LEFT(20), MID(20), RIGHT(10), RIGHT(20), SIMP(20), TRAP(20).

f is decreasing on $[0, 2]$ so

$$\text{RIGHT}(10) \leq \text{RIGHT}(2 \cdot 10) \leq \int_0^2 f(t)\,dt \leq \text{LEFT}(2 \cdot 10) \leq \text{LEFT}(10)$$

and

$$\text{RIGHT}(20) \leq \text{MID}(20) \leq \text{LEFT}(20)$$

and

$$\text{RIGHT}(20) \leq \text{TRAP}(20) \leq \text{LEFT}(20)$$

f is concave down on $[0, 2]$ so

$$\text{TRAP}(20) \leq \int_0^2 f(t)\,dt \leq \text{MID}(20)$$

and

$$\text{TRAP}(20) \leq \text{SIMP}(20) \leq \text{MID}(20)$$

Combining all these inequalities we get

$$\boxed{\text{RIGHT}(10) \leq \text{RIGHT}(20) \leq \text{TRAP}(20) \leq \text{SIMP}(20) \leq \text{MID}(20) \leq \text{LEFT}(20) \leq \text{LEFT}(10)}$$

6. For each of the following improper integrals, say whether it converges or diverges. If it converges, find its value.

$$\int_0^\infty \frac{1}{1+x^2}\,dx \qquad \int_2^\infty \frac{1}{x}\,dx \qquad \int_{0.5}^2 \frac{1}{1-x}\,dx \qquad \int_1^\infty \frac{1}{x^3}\,dx$$

$$\begin{aligned}
\int_0^\infty \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx \\
&= \lim_{b\to\infty} \arctan(x)|_0^b \\
&= \lim_{b\to\infty} \arctan(b) - \arctan(0) \\
&= \boxed{\tfrac{\pi}{2} \text{ convergent}}
\end{aligned}$$

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \ln|x|\,|_2^b$$
$$= \lim_{b\to\infty} \ln|b| - \ln(2)$$
$$\boxed{\text{divergent}}$$

$$\int_{0.5}^2 \frac{1}{1-x}\,dx = \int_{0.5}^1 \frac{1}{1-x}\,dx + \int_1^2 \frac{1}{1-x}\,dx$$
$$= \lim_{b\to 1^-}\int_{0.5}^b \frac{1}{1-x}\,dx + \lim_{c\to 1^+}\int_c^2 \frac{1}{1-x}\,dx$$
$$= \lim_{b\to 1^-}(-\ln|1-x|\,|_{0.5}^b) + \lim_{c\to 1^+}(-\ln|1-x|\,|_c^2)$$
$$= \lim_{b\to 1^-}(-\ln|1-b| + \ln|1-0.5|) + \lim_{c\to 1^+}(-\ln|1-2| + \ln|1-c|)$$
$$\boxed{\text{divergent}}$$

$$\int_1^\infty \frac{1}{x^3}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx$$
$$= \lim_{b\to\infty} \left.\frac{1}{-2x^2}\right|_1^b$$
$$= \lim_{b\to\infty} \frac{1}{-2b^2} - \frac{1}{-2\cdot 1^2}$$
$$= \boxed{\tfrac{1}{2}\ \text{convergent}}$$

7. A solid has its base in the (x, y)-plane. The base of the solid is the region between the curve $y = 4 - x^2$ and the x-axis. Each cross-section of the solid perpendicular to the x-axis is a square. Find the volume of the solid.

$$A(x) = y^2 = (4 - x^2)^2$$

The roots of $4 - x^2$ are $x = 2$ and $x = -2$ so

$$V = \int_{-2}^2 A(x)\,dx$$
$$= \int_{-2}^2 (4 - x^2)^2\,dx$$
$$= \int_{-2}^2 16 - 8x^2 + x^4\,dx$$
$$= 16x - \frac{8x^3}{3} + \left.\frac{x^5}{5}\right|_{-2}^2$$
$$= \left(16\cdot 2 - \frac{8\cdot 2^3}{3} + \frac{2^5}{5}\right) - \left(16\cdot(-2) - \frac{8\cdot(-2)^3}{3} + \frac{(-2)^5}{5}\right)$$
$$= \boxed{\tfrac{512}{15}}$$

8. Find the volume of the donut obtained by rotating a circle of radius 1 and center $(0, 2)$ around the x-axis.

$$\begin{aligned}
A(x) &= \pi R(x)^2 - \pi r(x)^2 \\
&= \pi(2+\sqrt{1-x^2})^2 - \pi(2-\sqrt{1-x^2})^2 \\
&= \pi(4+4\sqrt{1-x^2}+(1-x^2)) - \pi(4-4\sqrt{1-x^2}+(1-x^2)) \\
&= 8\pi\sqrt{1-x^2}
\end{aligned}$$

$$\begin{aligned}
V &= \int_{-1}^{1} A(x)\,dx \\
&= \int_{-1}^{1} 8\pi\sqrt{1-x^2}\,dx \\
&\text{(trig substitution } x=\sin\theta, dx=\cos\theta d\theta \text{ so } \theta=\arcsin x) \\
&= 8\pi\int_{\arcsin(-1)}^{\arcsin 1} (\sqrt{1-\sin^2\theta})\cos\theta\,d\theta \\
&= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sqrt{\cos^2\theta})\cos\theta\,d\theta \\
&= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\
&= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= 8\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}+\frac{1}{2}\cos(2\theta)\,d\theta \\
&= 8\pi\left(\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 8\pi\left(\frac{1}{2}\cdot\frac{\pi}{2}+\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)\right) - 8\pi\left(\frac{1}{2}\left(-\frac{\pi}{2}\right)+\frac{1}{4}\sin\left(2\cdot\left(-\frac{\pi}{2}\right)\right)\right) \\
&= \boxed{4\pi^2}
\end{aligned}$$

9. Suppose that $g(x)$ is a function such that $g(x) \geq 0$ and $g'(x) \geq 1$ for all $x \geq 0$. Define a function $f(x)$ by the formula $f(x) = \int_0^x \sqrt{(g'(t))^2 - 1}\,dt$. Show that the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to $g(b) - g(a)$.

$$\begin{aligned}
f(x) &= \int_0^x \sqrt{(g'(t))^2-1}\,dt \\
f'(x) &= \frac{d}{dx}\int_0^x \sqrt{(g'(t))^2-1}\,dt \\
&\text{(by the fundamental theorem of calculus)} \\
&= \sqrt{(g'(x))^2-1}
\end{aligned}$$

$$\begin{aligned}
\text{arc length of } f &= \int_a^b \sqrt{1+(f'(x))^2}\,dx \\
&\text{(by the calculation above)} \\
&= \int_a^b \sqrt{1+\left(\sqrt{(g'(x))^2-1}\right)^2}\,dx \\
&= \int_a^b \sqrt{1+(g'(x))^2-1}\,dx \\
&= \int_a^b \sqrt{(g'(x))^2}\,dx \\
&= \int_a^b g'(x)\,dx \\
&\text{(by the fundamental theorem of calculus)} \\
&= g(b)-g(a)
\end{aligned}$$

10. Find the arc length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ on the interval between the points $(0,1)$ and $(1,(e+1/e)/2)$.

$$\begin{aligned}
f(x) &= \frac{1}{2}(e^x + e^{-x}) \\
f'(x) &= \frac{1}{2}(e^x - e^{-x})
\end{aligned}$$

$$\begin{aligned}
\text{arc length} &= \int_0^1 \sqrt{1+(f'(x))^2}\,dx \\
&= \int_0^1 \sqrt{1+\left(\frac{1}{2}(e^x-e^{-x})\right)^2}\,dx \\
&= \int_0^1 \sqrt{1+\frac{1}{4}e^{2x}-\frac{1}{2}e^xe^{-x}+\frac{1}{4}e^{-2x}}\,dx \\
&= \int_0^1 \sqrt{\frac{1}{4}e^{2x}+\frac{1}{2}+\frac{1}{4}e^{-2x}}\,dx \\
&= \int_0^1 \sqrt{\left(\frac{1}{2}e^x+\frac{1}{2}e^{-x}\right)^2}\,dx \\
&= \int_0^1 \frac{1}{2}e^x+\frac{1}{2}e^{-x}\,dx \\
&= \frac{1}{2}e^x-\frac{1}{2}e^{-x}\Big|_0^1 \\
&= \left(\frac{1}{2}e^1-\frac{1}{2}e^{-1}\right)-\left(\frac{1}{2}e^0-\frac{1}{2}e^0\right) \\
&= \boxed{\tfrac{e}{2}-\tfrac{1}{2e}}
\end{aligned}$$

11. Find the arc length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln(x)$ between $x = 1$ and $x = 2$.

$$\begin{aligned}
f(x) &= \frac{1}{2}x^2 - \frac{1}{4}\ln(x) \\
f'(x) &= x - \frac{1}{4x}
\end{aligned}$$

$$
\begin{aligned}
\text{arc length} &= \int_1^2 \sqrt{1+(f'(x))^2}\,dx \\
&= \int_1^2 \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx \\
&= \int_1^2 \sqrt{1+x^2-\frac{1}{2}+\frac{1}{16x^2}}\,dx \\
&= \int_1^2 \sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}\,dx \\
&= \int_1^2 \sqrt{\left(x+\frac{1}{4x}\right)^2}\,dx \\
&= \int_1^2 x+\frac{1}{4x}\,dx \\
&= \frac{x^2}{2}+\frac{\ln x}{4}\Bigg|_1^2 \\
&= \left(\frac{2^2}{2}+\frac{\ln 2}{4}\right)-\left(\frac{1^2}{2}+\frac{\ln 1}{4}\right) \\
&= \boxed{\tfrac{3}{2}+\tfrac{\ln 2}{4}}
\end{aligned}
$$

12. Find the area of the region in the plane, which (in polar coordinates) is bounded by the lines $\theta = 0$, $\theta = \pi$ and by the portion of the logarithmic spiral $r = e^{\theta}$ for $0 \leq \theta \leq \pi$.

$$
\begin{aligned}
\text{Area} &= \int_0^{\pi} \frac{1}{2}(r(\theta))^2\,d\theta \\
&= \int_0^{\pi} \frac{1}{2}\left(e^{\theta}\right)^2\,d\theta \\
&= \frac{1}{2}\int_0^{\pi} e^{2\theta}\,d\theta \\
&= \frac{1}{4}e^{2\theta}\Bigg|_0^{\pi} \\
&= \frac{1}{4}e^{2\pi}-\frac{1}{4}e^{2\cdot 0} \\
&= \boxed{\tfrac{e^{2\pi}}{4}-\tfrac{1}{4}}
\end{aligned}
$$

13. Find the center of mass of a hemisphere of radius R whose base lies in the (x, y)-plane. (Hint: take the center of the hemisphere to lie at the origin. By symmetry you may assume that the center of mass lies on the positive z-axis.)

$$
\begin{aligned}
A(z) &= \pi(r(z))^2 \\
&= \pi(\sqrt{R^2-z^2})^2 \\
&= \pi(R^2-z^2)
\end{aligned}
$$

$$
\begin{aligned}
\bar{z} &= \frac{\int_0^R z\delta A(z)\,dz}{\int_0^R \delta A(z)\,dz} \\
&= \frac{\int_0^R z\delta\pi(R^2 - z^2)\,dz}{\int_0^R \delta\pi(R^2 - z^2)\,dz} \\
&= \frac{\delta\pi \int_0^R R^2 z - z^3\,dz}{\delta\pi \int_0^R R^2 - z^2\,dz} \\
&= \frac{\frac{R^2 z^2}{2} - \frac{z^4}{4}\Big|_0^R}{R^2 z - \frac{z^3}{3}\Big|_0^R} \\
&= \frac{\frac{R^4}{2} - \frac{R^4}{4}}{R^3 - \frac{R^3}{3}} \\
&= \frac{3}{8}R
\end{aligned}
$$

By symmetry $\bar{x} = 0$ and $\bar{y} = 0$.

The center of mass is the point $(0, 0, \frac{3}{8}R)$.

14. Let C_1 and C_2 be two long cylinders of radius one in space, so that the axis of symmetry of C_1 is the x-axis and the axis of symmetry of C_2 is the y-axis. Find the volume of the intersection of C_1 and C_2. (Hint: each cross-section perpendicular to the z-axis is a square.)

C_1 has equation $y^2 + z^2 = 1$ and C_2 has equation $x^2 + z^2 = 1$. For a fixed value of z the cylinders intersect in a square with sides of length $2y = 2x = 2\sqrt{1 - z^2}$.

$$
\begin{aligned}
A(z) &= (2\sqrt{1 - z^2})^2 \\
&= 4 - 4z^2
\end{aligned}
$$

$$
\begin{aligned}
V &= \int_{-1}^{1} A(z)\,dz \\
&= \int_{-1}^{1} 4 - 4z^2\,dz \\
&= 4z - \frac{4z^3}{3}\Big|_{-1}^{1} \\
&= \left(4 \cdot 1 - \frac{4 \cdot 1^3}{3}\right) - \left(4 \cdot (-1) - \frac{4 \cdot (-1)^3}{3}\right) \\
&= \boxed{\frac{16}{3}}
\end{aligned}
$$