

## Practice Midterm 1 - Solutions

1. (a) Suppose that  $f(x)$  is an even function and that  $\int_0^4 f(x)dx = 6$ , and that  $\int_{-2}^2 f(x)dx = 2$ . Find  $\int_2^4 f(x)dx$ .

$$\begin{aligned}\int_2^4 f(x)dx &= \int_0^4 f(x)dx - \int_0^2 f(x)dx \\ &= \int_0^4 f(x)dx - \frac{1}{2} \int_{-2}^2 f(x)dx \quad (\text{since } f \text{ is even}) \\ &= 6 - \frac{2}{2} \\ &= \boxed{5}\end{aligned}$$

- (b) Suppose that  $g(x)$  is an odd function and that  $\int_0^4 g(x)dx = 3$ , and  $\int_0^2 g(x)dx = 1$ . Find  $\int_{-2}^4 g(x)dx$ .

$$\begin{aligned}\int_{-2}^4 g(x)dx &= \int_{-2}^0 g(x)dx + \int_0^4 g(x)dx \\ &= -\int_0^2 g(x)dx + \int_0^4 g(x)dx \quad (\text{since } g \text{ is odd}) \\ &= -1 + 3 \\ &= \boxed{2}\end{aligned}$$

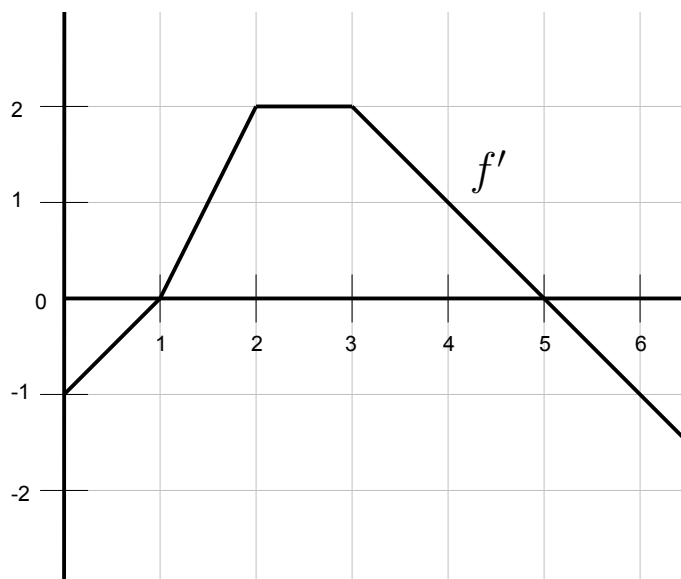
2. Draw an area that represents the definite integral  $\int_0^3 \sqrt{9-x^2} dx$ . Evaluate this integral.

This is a quarter of circle with radius 3 so

$$\int_0^3 \sqrt{9-x^2} dx = \pi 3^2 \cdot \frac{1}{4} = \boxed{\frac{9\pi}{4}}$$

3. Suppose  $f'(x)$  is given by the graph below, and that  $f(0) = \frac{3}{2}$ . Complete the table of values for  $f(x)$ .

$x$	0	1	3	5
$f(x)$	$\frac{3}{2}$	1	4	6



4. Find the following indefinite integrals.

(a)  $\int \pi x^2 + \frac{1}{\pi x^2} dx$

$$\begin{aligned} \int \pi x^2 + \frac{1}{\pi x^2} dx &= \int \pi x^2 + \frac{1}{\pi} \cdot x^{-2} dx \\ &= \frac{\pi x^3}{3} + \frac{x^{-1}}{\pi(-1)} + C \\ &= \boxed{\frac{\pi x^3}{3} - \frac{1}{\pi x} + C} \end{aligned}$$

(b)  $\int \sqrt{x} - \frac{1}{x\sqrt{x}} dx$

$$\begin{aligned} \int \sqrt{x} - \frac{1}{x\sqrt{x}} dx &= \int x^{\frac{1}{2}} - x^{-\frac{3}{2}} dx \\ &= \frac{2}{3} \cdot x^{\frac{3}{2}} - \left(-\frac{2}{1}\right) \cdot x^{-\frac{1}{2}} + C \\ &= \boxed{\frac{2}{3} \cdot x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + C} \end{aligned}$$

(c)  $\int e^t + e^3 dt$

$$\int e^t + e^3 dt = \boxed{e^t + e^3 t + C}$$

(d)  $\int \frac{2}{x} + \frac{x}{2} dx$

$$\begin{aligned} \int \frac{2}{x} + \frac{x}{2} dx &= \int 2x^{-1} + \frac{1}{2} \cdot x dx \\ &= 2 \ln x + \frac{1}{2} \cdot \frac{x^2}{2} + C \\ &= \boxed{2 \ln x + \frac{x^2}{4} + C} \end{aligned}$$

5. Write down the left and right Riemann sums for three equal intervals for the integral

$$\int_1^2 \frac{1}{x} dx.$$

Without using your calculator explain why

$$\frac{47}{60} \geq \ln(2) \geq \frac{37}{60}.$$

Let  $f(x) = \frac{1}{x}$

$$\begin{aligned} \text{LEFT}(3) &= \sum_{i=0}^{3-1} f\left(1 + i \cdot \frac{2-1}{3}\right) \cdot \frac{2-1}{3} \\ &= f(1) \cdot \frac{1}{3} + f\left(\frac{4}{3}\right) \cdot \frac{1}{3} + f\left(\frac{5}{3}\right) \cdot \frac{1}{3} \\ &= \frac{1}{1} \cdot \frac{1}{3} + \frac{3}{4} \cdot \frac{1}{3} + \frac{3}{5} \cdot \frac{1}{3} \\ &= \frac{20}{60} + \frac{15}{60} + \frac{12}{60} \\ &= \boxed{\frac{47}{60}} \end{aligned}$$

$$\begin{aligned} \text{RIGHT}(3) &= \sum_{i=1}^3 f\left(1 + i \cdot \frac{2-1}{3}\right) \cdot \frac{2-1}{3} \\ &= f\left(\frac{4}{3}\right) \cdot \frac{1}{3} + f\left(\frac{5}{3}\right) \cdot \frac{1}{3} + f(2) \cdot \frac{1}{3} \\ &= \frac{3}{4} \cdot \frac{1}{3} + \frac{3}{5} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{15}{60} + \frac{12}{60} + \frac{10}{60} \\ &= \boxed{\frac{37}{60}} \end{aligned}$$

$\frac{d}{dx} \ln x = \frac{1}{x}$  so by the Fundamental Theorem of Calculus  $\int_1^2 \frac{1}{x} dx = \ln 2 - \ln 1 = \ln 2$ . On the interval  $[1, 2]$  the function  $\frac{1}{x}$  is a decreasing so

$$\text{LEFT}(3) \geq \int_1^2 \frac{1}{x} dx \geq \text{RIGHT}(3)$$

hence

$$\frac{47}{60} \geq \ln(2) \geq \frac{37}{60}.$$

6. Use the Fundamental Theorem of Calculus to compute the following definite integrals.

(a)  $\int_1^3 \frac{1}{t^2} dt$

$$\begin{aligned}
\int_1^3 \frac{1}{t^2} dt &= \int_1^3 t^{-2} dt \\
&= \left. \frac{t^{-1}}{-1} \right|_1^3 \\
&= -\frac{1}{3} - \left( -\frac{1}{1} \right) \\
&= \boxed{\frac{2}{3}}
\end{aligned}$$

(b)  $\int_0^\pi (3 \sin x + x + 5) dx$

$$\begin{aligned}
\int_0^\pi (3 \sin x + x + 5) dx &= -3 \cos x + \frac{x^2}{2} + 5x \Big|_0^\pi \\
&= \left( -3 \cos \pi + \frac{\pi^2}{2} + 5\pi \right) - \left( -3 \cos 0 + \frac{0^2}{2} + 5 \cdot 0 \right) \\
&= 3 + \frac{\pi^2}{2} + 5\pi + 3 \\
&= \boxed{6 + \frac{\pi^2}{2} + 5\pi}
\end{aligned}$$

7. Find the derivative of  $x \ln(x) - x$ . Calculate  $\int_2^3 \ln(x) dx$ .

$$\frac{d}{dx} (x \ln(x) - x) = 1 \cdot \ln x + x \cdot \frac{1}{x} - 1 = \boxed{\ln x}$$

By The Fundamental Theorem of Calculus and the differentiation above

$$\int_2^3 \ln(x) dx = (x \ln(x) - x) \Big|_2^3 = (3 \ln(3) - 3) - (2 \ln(2) - 2) = \boxed{3 \ln(3) - 2 \ln(2) + 1}$$

8. Find the derivative of  $G(x) = \int_{\sin x}^{x^3} t \sqrt{t^4 + 1} dt$

$$\begin{aligned}
\frac{d}{dx} G(x) &= \frac{d}{dx} \left( \int_{\sin x}^{x^3} t \sqrt{t^4 + 1} dt \right) \\
&= \frac{d}{dx} \left( \int_{\sin x}^0 t \sqrt{t^4 + 1} dt + \int_0^{x^3} t \sqrt{t^4 + 1} dt \right) \\
&= \frac{d}{dx} \left( - \int_0^{\sin x} t \sqrt{t^4 + 1} dt + \int_0^{x^3} t \sqrt{t^4 + 1} dt \right) \\
&= \boxed{-\sin x \sqrt{(\sin x)^4 + 1} \cdot (\cos x) + x^3 \sqrt{(x^3)^4 + 1} \cdot 3x^2}
\end{aligned}$$

9. The average value of a function  $f(x)$  for  $1 \leq x \leq 5$  is equal to 7, and the average value of the same function  $f(x)$  for  $5 \leq x \leq 8$  is equal to 3. What is the average value of  $f(x)$  for  $1 \leq x \leq 8$ ?

$$\frac{1}{5-1} \int_1^5 f(x) dx = 7 \text{ so } \int_1^5 f(x) dx = 7 \cdot 4 = 28.$$

$$\frac{1}{8-5} \int_5^8 f(x) dx = 3 \text{ so } \int_5^8 f(x) dx = 3 \cdot 3 = 9.$$

Average value of  $f$  on  $[1, 8]$  is

$$\begin{aligned} \frac{1}{8-1} \int_1^8 f(x) dx &= \frac{1}{8-1} \left( \int_1^5 f(x) dx + \int_5^8 f(x) dx \right) \\ &= \frac{1}{7} (28 + 9) \\ &= \boxed{\frac{37}{7}} \end{aligned}$$