**Mathematics 152.02**
**Review for Midterm 2**

1. The degree of the numerator is greater than or equal (in this case strictly greater) than the degree of the denominator. So start by dividing through:

$$x^3 + 4x + 1 = x(x^2 + 3) + x + 1,$$

and so

$$f'(x) = x + \frac{x+1}{x^2+3} = x + \frac{x}{x^2+3} + \frac{1}{x^2+3}.$$

Integrating each of these terms separately (noting that the derivative of $x^2 + 3$ is $2x$), we get

$$f(x) = \frac{x^2}{2} + \frac{1}{2}\ln(x^2+3) + \frac{1}{\sqrt{3}}\arctan(\frac{x}{\sqrt{3}}) + C.$$

Finally, discover that $C = 4 - \frac{3}{2}\ln(3)$ by using the equation $f(0) = 4$.

2. The case when $a = 0$ but $b \neq 0$ and the case when $b = 0$ but $a \neq 0$ are easy, so we just need to do the case when $a$ and $b$ are both non-zero. In this case, use integration by parts twice. You can use the parts either way around, as long as you choose the same way both times. I took $u = e^{ax}$, $v' = \cos(bx)$, and so $v = \frac{1}{b}\sin(bx)$. This gave

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{b}e^{ax}\sin(bx) - \frac{a}{b}\int e^{ax}\sin(bx)\,dx.$$

Now use parts again, taking $u_1 = e^{ax}$ and $v_1' = \sin(bx)$, so that $v_1 = -\frac{1}{b}\cos(bx)$. This gives

$$\int e^{ax}\cos(bx)\,dx = \frac{1}{b}e^{ax}\sin(bx) + \frac{a}{b^2}e^{ax}\cos(bx) - \frac{a^2}{b^2}\int e^{ax}\cos(bx)\,dx.$$

Simplifying this equation gives

$$(a^2 + b^2)\int e^{ax}\cos(bx)\,dx = e^{ax}(a\cos(bx) + b\sin(bx)) + C$$

and hence the claimed result.

3. Write $g(x) = \int_c^x \frac{t}{1+e^t}\,dt$ for any constant $c$. Then $g'(x) = \frac{x}{1+e^x}$, and $f(x) = g(x^3) - g(x^{1/2})$. The chain rule now gives

$$f'(x) = 3x^2 g'(x^3) - \frac{1}{2}x^{-1/2}g'(x^{1/2}) = \frac{3x^5}{1+e^{x^3}} - \frac{1}{2(1+e^{\sqrt{x}})}.$$

4. This is a portion of a circle of radius 2. It can be split into a pie-piece of size $1/12$th of the whole circle, together with a right-angled triangle with side lengths $1$, $\sqrt{3}$ and 2. This gives an area of $\pi/3 + \sqrt{3}/2$. But the integration way to attack this question is as

$$\int_{x=0}^{x=1} \sqrt{4-x^2}\, dx.$$

Substitute $x = 2\sin(\theta)$, which gives $dx = 2\cos(\theta) d\theta$. The integral becomes

$$\int_{x=0}^{x=1} \sqrt{4-x^2}\, dx = \int_{\theta=0}^{\theta=\pi/6} 4\cos^2(\theta)\, d\theta.$$

One can use the double angle formula $\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))$ to do this integral, obtaining the answer given earlier.

5. Integrate the equation

$$a(t) = 1 + 2t + t^2,$$

to obtain

$$v(t) = t + t^2 + \frac{1}{3}t^3 + C.$$

When $t = 0$, $v(t) = 0$, and so $C = 0$. Now integrate the equation

$$v(t) = t + t^2 + \frac{1}{3}t^3$$

to obtain

$$s(t) = \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4 + C.$$

Again, the initial value $s(0) = 0$ tells us that $C = 0$, and so

$$s(t) = \frac{1}{2}t^2 + \frac{1}{3}t^3 + \frac{1}{12}t^4.$$

6. You are asked to check that $l(ab) = l(a) + l(b)$, using only the given definition of $l(x)$ as an integral. This amounts to checking the equation

$$\int_1^{ab} \frac{1}{t}\, dt = \int_1^{a} \frac{1}{t}\, dt + \int_1^{b} \frac{1}{t}\, dt.$$

By additivity of the integral over its range we see that

$$\int_{t=1}^{t=ab} \frac{1}{t}\, dt = \int_{t=1}^{t=a} \frac{1}{t}\, dt + \int_{t=a}^{t=ab} \frac{1}{t}\, dt.$$

Now in the second integral on the right-hand side, substitute $u = at$, which gives $du = adt$. This gives

$$\int_{t=1}^{t=ab} \frac{1}{t}\, dt = \int_{t=1}^{t=a} \frac{1}{t}\, dt + \int_{u=1}^{u=b} \frac{1}{u}\, du,$$

and so

$$\int_{t=1}^{t=ab} \frac{1}{t}\, dt = \int_{t=1}^{t=a} \frac{1}{t}\, dt + \int_{t=1}^{t=b} \frac{1}{t}\, dt,$$

as claimed.

7. First we do the indefinite integral by using parts twice:

$$\int t^2e^{-t}\,dt = -t^2e^{-t}+\int 2te^{-t}\,dt = -t^2e^{-t}-2te^{-t}+\int 2e^{-t}\,dt = -t^2e^{-t}-2te^{-t}-2e^{-t}+C.$$

Hence we see that

$$\int_1^\infty t^2e^{-t}\,dt = \lim_{b\to\infty}(-b^2e^{-b}-2be^{-b}-2e^{-b}+e^{-1}+2e^{-1}+2e^{-1}).$$

Since $P(b)e^{-b}$ tends to 0 as $b\to\infty$ for any polynomial $P(b)$, we see that the integral converges to $5e^{-1}$.

8. (a) is true: the integrand, $\sin(1-x^2)$ is positive for all $x\leq 1$, and so since $f(0.5)$ is the integral of a positive function it must be positive.

   (b) is also true. By the fundamental theorem of calculus, $f'(x)=\sin(1-x^2)$, and so $f'(1)=0$ so there is a critical point at $x=1$. The second derivative $f''(x)$ is equal to $-2x\cos(1-x^2)$, so $f''(1)=-2$ is strictly negative, which implies that $x=1$ is a local maximum for $f(x)$.

   (c) is again true. Since $1-2^2=-3$ is close to $-\pi$, we see that $f'(-3)=\sin(-3)$ must be close to $-1$, and so $f(x)$ is decreasing near $x=2$.

   (d) is false, although it is hard to work this one out without using a calculator. Here $f''(3)=-6\cos(-8)>0$, so near $x=3$ the function is concave up.