Mathematics 152.02
Solutions to Review for Final Exam

1. Suppose that the function $f(t)$ represents the price of a barrel of oil in dollars at time t, measured in days, where $f(0)$ is the price at midnight on December 31, 2005. In words, what is represented by $\frac{1}{90}\int_0^{90} f(t)\,dt$?

$\frac{1}{90}\int_0^{90} f(t)\,dt$ is the average price of a barrel of oil in dollars during the first 90 days of 2006.

2. Suppose that $f(x)$ is an odd function, $g(x)$ is an even function, and that the following values are known:

$$\int_0^3 f(x)\,dx = 2 \qquad \int_3^5 f(x)\,dx = 1 \qquad \int_0^3 g(x)\,dx = 4 \qquad \int_3^5 g(x)\,dx = 1.$$

Find $\int_{-3}^5 f(x) + g(x)\,dx$.

$$\begin{aligned}
\int_{-3}^5 f(x) + g(x)\,dx &= \int_{-3}^5 f(x)\,dx + \int_{-3}^5 g(x)\,dx \\
&= \int_{-3}^3 f(x)\,dx + \int_3^5 f(x)\,dx + \int_{-3}^3 g(x)\,dx + \int_3^5 f(x)\,dx \\
&= 0 + \int_3^5 f(x)\,dx + 2\int_0^3 g(x)\,dx + \int_3^5 f(x)\,dx \\
&= 0 + 1 + 2 \cdot 4 + 1 \\
&= \boxed{10}
\end{aligned}$$

3. Find the solution to the initial value problem

$$f'(t) = \frac{t^3+1}{t^3-4t} \qquad f(1) = 2$$

$$\begin{aligned}
f'(t) &= \frac{t^3+1}{t^3-4t} \\
&= 1 + \frac{4t+1}{t^3-4t} \\
&= 1 + \frac{4t+1}{t(t+2)(t-2)} \\
&= 1 + \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-2} \\
&= 1 - \frac{1}{4t} - \frac{7}{8(t+2)} + \frac{9}{8(t-2)}
\end{aligned}$$

Thus

$$\begin{aligned}
f(t) &= \int 1 - \frac{1}{4t} - \frac{7}{8(t+2)} + \frac{9}{8(t-2)}\,dx \\
&= t - \frac{\ln|t|}{4} - \frac{7\ln|t+2|}{8} + \frac{9\ln|t-2|}{8} + C
\end{aligned}$$

$f(1) = 2$ implies that

$$\begin{aligned}
2 &= 1 - \frac{\ln|1|}{4} - \frac{7\ln|1+2|}{8} + \frac{9\ln|1-2|}{8} + C \\
C &= 2 - 1 + \frac{\ln|1|}{4} + \frac{7\ln|1+2|}{8} - \frac{9\ln|1-2|}{8} \\
C &= 2 - 1 + \frac{0}{4} + \frac{7\ln 3}{8} - \frac{9\cdot 0}{8} \\
C &= 1 + \frac{7\ln 3}{8}
\end{aligned}$$

$$\boxed{f(t) = t - \frac{\ln|t|}{4} - \frac{7\ln|t+2|}{8} + \frac{9\ln|t-2|}{8} + 1 + \frac{7\ln 3}{8}}$$

4. For each of the following improper integrals, say whether it converges or diverges. If it converges, find its value.

$$\int_0^\infty \frac{1}{1+x^2}\,dx \qquad \int_2^\infty \frac{1}{x}\,dx \qquad \int_{0.5}^2 \frac{1}{1-x}\,dx \qquad \int_1^\infty \frac{1}{x^3}\,dx$$

$$\begin{aligned}\int_0^\infty \frac{1}{1+x^2}\,dx &= \lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\,dx \\ &= \lim_{b\to\infty} \arctan(x)|_0^b \\ &= \lim_{b\to\infty} \arctan(b) - \arctan(0) \\ &= \boxed{\tfrac{\pi}{2}\text{ convergent}}\end{aligned}$$

$$\begin{aligned}\int_2^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty} \int_2^b \frac{1}{x}\,dx \\ &= \lim_{b\to\infty} \ln|x|\,|_2^b \\ &= \lim_{b\to\infty} \ln|b| - \ln(2) \\ &\boxed{\text{divergent}}\end{aligned}$$

$$\begin{aligned}\int_{0.5}^2 \frac{1}{1-x}\,dx &= \int_{0.5}^1 \frac{1}{1-x}\,dx + \int_1^2 \frac{1}{1-x}\,dx \\ &= \lim_{b\to1^-} \int_{0.5}^b \frac{1}{1-x}\,dx + \lim_{c\to1^+} \int_c^2 \frac{1}{1-x}\,dx \\ &= \lim_{b\to1^-} (-\ln|1-x|\,|_{0.5}^b) + \lim_{c\to1^+} (-\ln|1-x|\,|_c^2) \\ &= \lim_{b\to1^-} (-\ln|1-b| + \ln|1-0.5|) + \lim_{c\to1^+} (-\ln|1-2| + \ln|1-c|) \\ &\boxed{\text{divergent}}\end{aligned}$$

$$\begin{aligned}\int_1^\infty \frac{1}{x^3}\,dx &= \lim_{b\to\infty} \int_1^b \frac{1}{x^3}\,dx \\ &= \lim_{b\to\infty} \left.\frac{1}{-2x^2}\right|_1^b \\ &= \lim_{b\to\infty} \frac{1}{-2b^2} - \frac{1}{-2\cdot1^2} \\ &= \boxed{\tfrac{1}{2}\text{ convergent}}\end{aligned}$$

5. A solid has its base in the (x, y)-plane. The base of the solid is the region between the curve $y = 4 - x^2$ and the x-axis. Each cross-section of the solid perpendicular to the x-axis is a square. Find the volume of the solid.

$$A(x) = y^2 = (4 - x^2)^2$$

The roots of $4 - x^2$ are $x = 2$ and $x = -2$ so

$$\begin{aligned}
V &= \int_{-2}^{2} A(x)\,dx \\
&= \int_{-2}^{2} (4 - x^2)^2\,dx \\
&= \int_{-2}^{2} 16 - 8x^2 + x^4\,dx \\
&= 16x - \frac{8x^3}{3} + \frac{x^5}{5}\Big|_{-2}^{2} \\
&= \left(16 \cdot 2 - \frac{8 \cdot 2^3}{3} + \frac{2^5}{5}\right) - \left(16 \cdot (-2) - \frac{8 \cdot (-2)^3}{3} + \frac{(-2)^5}{5}\right) \\
&= \boxed{\tfrac{512}{15}}
\end{aligned}$$

6. Find the volume of the donut obtained by rotating a circle of radius 1 and center $(0, 2)$ around the x-axis.

$$\begin{aligned}
A(x) &= \pi R(x)^2 - \pi r(x)^2 \\
&= \pi(2 + \sqrt{1 - x^2})^2 - \pi(2 - \sqrt{1 - x^2})^2 \\
&= \pi(4 + 4\sqrt{1 - x^2} + (1 - x^2)) - \pi(4 - 4\sqrt{1 - x^2} + (1 - x^2)) \\
&= 8\pi\sqrt{1 - x^2}
\end{aligned}$$

$$\begin{aligned}
V &= \int_{-1}^{1} A(x)\,dx \\
&= \int_{-1}^{1} 8\pi\sqrt{1 - x^2}\,dx
\end{aligned}$$

(trig substitution $x = \sin\theta, dx = \cos\theta d\theta$ so $\theta = \arcsin x$)

$$\begin{aligned}
&= 8\pi \int_{\arcsin(-1)}^{\arcsin 1} (\sqrt{1 - \sin^2\theta}) \cos\theta\,d\theta \\
&= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sqrt{\cos^2\theta}) \cos\theta\,d\theta \\
&= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta \\
&= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos(2\theta)}{2}\,d\theta \\
&= 8\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} + \frac{1}{2}\cos(2\theta)\,d\theta \\
&= 8\pi \left(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 8\pi \left(\frac{1}{2} \cdot \frac{\pi}{2} + \frac{1}{4}\sin\left(2 \cdot \frac{\pi}{2}\right)\right) - 8\pi\left(\frac{1}{2}\left(-\frac{\pi}{2}\right) + \frac{1}{4}\sin\left(2 \cdot \left(-\frac{\pi}{2}\right)\right)\right) \\
&= \boxed{4\pi^2}
\end{aligned}$$

7. Suppose that $g(x)$ is a function such that $g(x) \geq 0$ and $g'(x) \geq 1$ for all $x \geq 0$. Define a function $f(x)$ by the formula $f(x) = \int_0^x \sqrt{(g'(t))^2 - 1}\,dt$. Show that the arc length of the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to $g(b) - g(a)$.

$$
\begin{aligned}
f(x) &= \int_0^x \sqrt{(g'(t))^2 - 1}\,dt \\
f'(x) &= \frac{d}{dx}\int_0^x \sqrt{(g'(t))^2 - 1}\,dt \\
&\quad \text{(by the fundamental theorem of calculus)} \\
&= \sqrt{(g'(x))^2 - 1}
\end{aligned}
$$

$$
\begin{aligned}
\text{arc length of } f &= \int_a^b \sqrt{1 + (f'(x))^2}\,dx \\
&\quad \text{(by the calculation above)} \\
&= \int_a^b \sqrt{1 + \left(\sqrt{(g'(x))^2 - 1}\right)^2}\,dx \\
&= \int_a^b \sqrt{1 + (g'(x))^2 - 1}\,dx \\
&= \int_a^b \sqrt{(g'(x))^2}\,dx \\
&= \int_a^b g'(x)\,dx \\
&\quad \text{(by the fundamental theorem of calculus)} \\
&= g(b) - g(a)
\end{aligned}
$$

8. Find the arc length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ on the interval between the points $(0, 1)$ and $(1, (e + 1/e)/2)$.

$$
\begin{aligned}
f(x) &= \frac{1}{2}(e^x + e^{-x}) \\
f'(x) &= \frac{1}{2}(e^x - e^{-x})
\end{aligned}
$$

$$
\begin{aligned}
\text{arc length} &= \int_0^1 \sqrt{1 + (f'(x))^2}\,dx \\
&= \int_0^1 \sqrt{1 + \left(\frac{1}{2}(e^x - e^{-x})\right)^2}\,dx \\
&= \int_0^1 \sqrt{1 + \frac{1}{4}e^{2x} - \frac{1}{2}e^x e^{-x} + \frac{1}{4}e^{-2x}}\,dx \\
&= \int_0^1 \sqrt{\frac{1}{4}e^{2x} + \frac{1}{2} + \frac{1}{4}e^{-2x}}\,dx \\
&= \int_0^1 \sqrt{\left(\frac{1}{2}e^x + \frac{1}{2}e^{-x}\right)^2}\,dx \\
&= \int_0^1 \frac{1}{2}e^x + \frac{1}{2}e^{-x}\,dx \\
&= \frac{1}{2}e^x - \frac{1}{2}e^{-x}\Big|_0^1 \\
&= \left(\frac{1}{2}e^1 - \frac{1}{2}e^{-1}\right) - \left(\frac{1}{2}e^0 - \frac{1}{2}e^0\right) \\
&= \boxed{\tfrac{e}{2} - \tfrac{1}{2e}}
\end{aligned}
$$

9. Find the arc length of the curve $y = \frac{1}{2}x^2 - \frac{1}{4}\ln(x)$ between $x = 1$ and $x = 2$.

$$\begin{aligned} f(x) &= \frac{1}{2}x^2 - \frac{1}{4}\ln(x) \\ f'(x) &= x - \frac{1}{4x} \end{aligned}$$

$$\begin{aligned} \text{arc length} &= \int_1^2 \sqrt{1 + (f'(x))^2}\, dx \\ &= \int_1^2 \sqrt{1 + \left(x - \frac{1}{4x}\right)^2}\, dx \\ &= \int_1^2 \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}}\, dx \\ &= \int_1^2 \sqrt{x^2 + \frac{1}{2} + \frac{1}{16x^2}}\, dx \\ &= \int_1^2 \sqrt{\left(x + \frac{1}{4x}\right)^2}\, dx \\ &= \int_1^2 x + \frac{1}{4x}\, dx \\ &= \frac{x^2}{2} + \frac{\ln x}{4}\Big|_1^2 \\ &= \left(\frac{2^2}{2} + \frac{\ln 2}{4}\right) - \left(\frac{1^2}{2} + \frac{\ln 1}{4}\right) \\ &= \boxed{\tfrac{3}{2} + \tfrac{\ln 2}{4}} \end{aligned}$$

10. Find the area of the region in the plane, which (in polar coordinates) is bounded by the lines $\theta = 0$, $\theta = \pi$ and by the portion of the logarithmic spiral $r = e^\theta$ for $0 \le \theta \le \pi$.

$$\begin{aligned} \text{Area} &= \int_0^\pi \frac{1}{2}(r(\theta))^2\, d\theta \\ &= \int_0^\pi \frac{1}{2}\left(e^\theta\right)^2\, d\theta \\ &= \frac{1}{2}\int_0^\pi e^{2\theta}\, d\theta \\ &= \frac{1}{4}e^{2\theta}\Big|_0^\pi \\ &= \frac{1}{4}e^{2\pi} - \frac{1}{4}e^{2\cdot 0} \\ &= \boxed{\tfrac{e^{2\pi}}{4} - \tfrac{1}{4}} \end{aligned}$$

11. Find the center of mass of a hemisphere of radius R whose base lies in the (x, y)-plane. (Hint: take the center of the hemisphere to lie at the origin. By symmetry you may assume that the center of mass lies on the positive z-axis.)

$$\begin{aligned} A(z) &= \pi(r(z))^2 \\ &= \pi(\sqrt{R^2 - z^2})^2 \\ &= \pi(R^2 - z^2) \end{aligned}$$

$$\begin{aligned}\bar{z} &= \frac{\int_0^R z\delta A(z)\,dz}{\int_0^R \delta A(z)\,dz}\\ &= \frac{\int_0^R z\delta\pi(R^2-z^2)\,dz}{\int_0^R \delta\pi(R^2-z^2)\,dz}\\ &= \frac{\delta\pi\int_0^R R^2z-z^3\,dz}{\delta\pi\int_0^R R^2-z^2\,dz}\\ &= \frac{\frac{R^2z^2}{2}-\frac{z^4}{4}\Big|_0^R}{R^2z-\frac{z^3}{3}\Big|_0^R}\\ &= \frac{\frac{R^4}{2}-\frac{R^4}{4}}{R^3-\frac{R^3}{3}}\\ &= \frac{3}{8}R\end{aligned}$$

By symmetry $\bar{x}=0$ and $\bar{y}=0$.

$\boxed{\text{The center of mass is the point } (0,0,\tfrac{3}{8}R)}$.

12. Let C_1 and C_2 be two long cylinders of radius one in space, so that the axis of symmetry of C_1 is the x-axis and the axis of symmetry of C_2 is the y-axis. Find the volume of the intersection of C_1 and C_2. (Hint: each cross-section perpendicular to the z-axis is a square.)

C_1 has equation $y^2+z^2=1$ and C_2 has equation $x^2+z^2=1$. For a fixed value of z the cylinders intersect in a square with sides of length $2y=2x=2\sqrt{1-z^2}$.

$$\begin{aligned}A(z) &= (2\sqrt{1-z^2})^2\\ &= 4-4z^2\end{aligned}$$

$$\begin{aligned}V &= \int_{-1}^{1} A(z)\,dz\\ &= \int_{-1}^{1} 4-4z^2\,dz\\ &= 4z-\frac{4z^3}{3}\Big|_{-1}^{1}\\ &= \left(4\cdot 1-\frac{4\cdot 1^3}{3}\right)-\left(4\cdot(-1)-\frac{4\cdot(-1)^3}{3}\right)\\ &= \boxed{\tfrac{16}{3}}\end{aligned}$$