Math 152.02
Calculus with Analytic Geometry II

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Velocity and distance

Constant velocity

If an object has

constant velocity v

and travels for

time t

then it travels a distance

$$d = vt$$

Example 1 (Estimate total distance)

Suppose you are driving for 3 hours and every hour you look at your speedometer

time (h)	0	1	2	3
velocity (km/h)	0	60	70	70

About how far did you travel?

Solution

We assume velocity is constant on each hour so we can use Constant velocity formula.

- Distance traveled in hour 1? $60 \cdot 1$
- Distance traveled in hour 2? $70 \cdot 1$
- Distance traveled in hour 3? $70 \cdot 1$

Total distance is about $60 \cdot 1 + 70 \cdot 1 + 70 \cdot 1 = 200\,\mathrm{km}$

How can we improve our estimate?

Example 2 (Better estimate total distance)

Suppose you check your speedometer every 0.5 hours.

time (h)	0	0.5	1	1.5	2	2.5	3
velocity (km/h)	0	30	60	80	70	60	70

Solution

Now we assume velocity is constant on each 1/2 hour so we can use Constant Velocity formula.

- Distance traveled in first 1/2 hour? $30 \cdot 0.5$
- Distance traveled in second 1/2 hour? $60 \cdot 0.5$ *etc.*

Total distance is about
$30 \cdot 0.5 + 60 \cdot 0.5 + 80 \cdot 0.5 + 70 \cdot 0.5 + 60 \cdot 0.5 + 70 \cdot 0.5 = 185\,\mathrm{km}$

Philosophy of Calculus

- Estimate a quantity
- Figure out how to improve estimate
- Take limit

Example 3 (Derivative)

Slope of tangent line to f at a.

If x is close to a then a good estimate is slope of secant line through $(a, f(a))$ and $(x, f(x))$

$$\frac{f(a) - f(x)}{a - x}$$

exact slope of tangent is limit

$$f'(a) = \lim_{x \to a} \frac{f(a) - f(x)}{a - x}$$

Example 4 (Exact distance traveled)

Let's apply this philosophy to distance traveled problem.

- Velocity at time t is $v(t)$
- Look at speedometer n times (equally spaced)
- Take limit as n goes to ∞

Let $\Delta t = \frac{3}{n}$

Approx. distance traveled is

$$v(\Delta t) \cdot \Delta t + v(2\Delta t) \cdot \Delta t + \cdots + v(n\Delta t) \cdot \Delta t$$
$$= v(\tfrac{3}{n}) \cdot \tfrac{3}{n} + v(2 \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n} + \cdots + v(n \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n}$$

Exact distance traveled is

$$\lim_{n\to\infty} \left(v(\tfrac{3}{n}) \cdot \tfrac{3}{n} + v(2 \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n} + \cdots + v(n \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n} \right)$$

It's annoying to write out long sums like

$$v(\tfrac{3}{n}) \cdot \tfrac{3}{n} + v(2 \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n} + \cdots + v(n \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n}$$

term by term.

Sigma Notation

$$\sum_{i=1}^{n} a_i$$

means

$$a_1 + a_2 + a_3 + \cdots + a_n$$

"the sum from i equals 1 to n of a sub i"

Example 5

Write the sum of the odd numbers between 1 and 21 in sigma notation.

$$\sum_{i=0}^{10} 2i + 1$$

Example 6

Write the sum

$$= v(\tfrac{3}{n}) \cdot \tfrac{3}{n} + v(2 \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n} + \cdots + v(n \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n}$$

in sigma notation

$$\sum_{i=1}^{n} v(i \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n}$$

So the exact distance traveled in Example 4 is

$$\lim_{n\to\infty} \sum_{i=1}^{n} v(i \cdot \tfrac{3}{n}) \cdot \tfrac{3}{n}$$

Left and right sums

There was an ambiguity as to what constant velocity to assume when we estimated distance traveled

time (h)	0	0.5	1	1.5	2	2.5	3
velocity (km/h)	0	30	60	80	70	60	70

Definition 7 (Left and right sums)

Let f be a function on the closed interval $[a, b]$

$$\text{LEFT}(n) = \sum_{i=0}^{n-1} f\left(a + i \cdot \tfrac{b-a}{n}\right) \cdot \tfrac{b-a}{n}$$

$$\text{RIGHT}(n) = \sum_{i=1}^{n} f\left(a + i \cdot \tfrac{b-a}{n}\right) \cdot \tfrac{b-a}{n}$$

Problem 8

time (h)	*0*	*0.5*	*1*	*1.5*	*2*	*2.5*	*3*
velocity (km/h)	*0*	*30*	*60*	*80*	*70*	*60*	*70*

For the velocities in the above table estimate the total distance traveled using

1. LEFT(2)
2. RIGHT(3)
3. RIGHT(6)

time (h)	0	0.5	1	1.5	2	2.5	3
velocity (km/h)	0	30	60	80	70	60	70

Solution to Problem 8

1.

$$
\begin{aligned}
\text{LEFT}(2) &= \sum_{i=0}^{2-1} v\left(0 + i \cdot \tfrac{3-0}{2}\right) \cdot \tfrac{3-0}{2} \\
&= v\left(0 + 0 \cdot \tfrac{3}{2}\right) \cdot \tfrac{3}{2} + v\left(0 + 1 \cdot \tfrac{3}{2}\right) \cdot \tfrac{3}{2} \\
&= v(0) \cdot \tfrac{3}{2} + v(1.5) \cdot \tfrac{3}{2} \\
&= 0 \cdot \tfrac{3}{2} + 80 \cdot \tfrac{3}{2} \\
&= \boxed{120\,\mathrm{km}}
\end{aligned}
$$

time (h)	0	0.5	1	1.5	2	2.5	3
velocity (km/h)	0	30	60	80	70	60	70

Solution to Problem 8

2.

$$\begin{aligned}
\text{RIGHT}(3) &= \sum_{i=1}^{3} v\left(0 + i \cdot \tfrac{3-0}{3}\right) \cdot \tfrac{3-0}{3} \\
&= v(0 + 1 \cdot 1) \cdot 1 + v(0 + 2 \cdot 1) \cdot 1 + v(0 + 3 \cdot 1) \cdot 1 \\
&= v(1) + v(2) + v(3) \\
&= 60 + 70 + 70 \\
&= \boxed{200\,\mathrm{km}}
\end{aligned}$$

time (h)	0	0.5	1	1.5	2	2.5	3
velocity (km/h)	0	30	60	80	70	60	70

Solution to Problem 8

3.

$$\begin{aligned}
\text{LEFT}(6) &= \sum_{i=0}^{6-1} v\left(0 + i \cdot \tfrac{3-0}{6}\right) \cdot \tfrac{3-0}{6} \\
&= v\left(0 + 0 \cdot \tfrac{3}{6}\right) \cdot \tfrac{3}{6} + v\left(0 + 1 \cdot \tfrac{3}{6}\right) \cdot \tfrac{3}{6} \\
&\quad + v\left(0 + 2 \cdot \tfrac{3}{6}\right) \cdot \tfrac{3}{6} + v\left(0 + 3 \cdot \tfrac{3}{6}\right) \cdot \tfrac{3}{6} \\
&\quad + v\left(0 + 4 \cdot \tfrac{3}{6}\right) \cdot \tfrac{3}{6} + v\left(0 + 5 \cdot \tfrac{3}{6}\right) \cdot \tfrac{3}{6} \\
&= v(0) \cdot \tfrac{1}{2} + v(0.5) \cdot \tfrac{1}{2} + v(1) \cdot \tfrac{1}{2} + v(1.5) \cdot \tfrac{1}{2} \\
&\quad + v(2) \cdot \tfrac{1}{2} + v(2.5) \cdot \tfrac{1}{2} \\
&= 0 \cdot 0.5 + 30 \cdot 0.5 + 60 \cdot 0.5 + 80 \cdot 0.5 \\
&\quad + 70 \cdot 0.5 + 60 \cdot 0.5 \\
&= \boxed{150\,\mathrm{km}}
\end{aligned}$$

The definite integral

Disclaimer: There are complicated functions which cannot be integrated, but all functions in this course are integrable (for more info take real analysis)

Definition 9 (Definite Integral)

Let f be a function on the interval $[a, b]$. The **definite integral** of f from a to b is

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i \cdot \Delta t\right) \cdot \Delta t$$

where $\Delta t = \frac{b-a}{n}$.

a and b are called the **limits of integration**.

Theorem 10

$$\int_a^b f(t)\,\mathrm{d}t = \lim_{n\to\infty} \mathrm{RIGHT}(n) = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(a + i \cdot \Delta t\right) \cdot \Delta t$$

$$= \lim_{n\to\infty} \mathrm{LEFT}(n) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f\left(a + i \cdot \Delta t\right) \cdot \Delta t$$

Estimates and bounds

Since $\int_a^b f(t)\,dt$ is a limit of left and right sums we can use left and right sums to estimate $\int_a^b f(t)\,dt$.

Problem 11

Estimate $\int_2^6 t^3\,dt$ *using* LEFT(4)

Solution to Problem 11

$$\begin{aligned}\text{LEFT}(4) &= \sum_{i=0}^{4-1} f\left(2 + i \cdot \tfrac{6-2}{4}\right) \cdot \tfrac{6-2}{4} \\ &= f(2+0\cdot 1)\cdot 1 + f(2+1\cdot 1)\cdot 1 \\ &\quad + f(2+2\cdot 1)\cdot 1 + f(2+3\cdot 1)\cdot 1 \\ &= f(2)+f(3)+f(4)+f(5) \\ &= 2^3+3^3+4^3+5^3 \\ &= \boxed{224}\end{aligned}$$

Problem 12

Estimate $\int_2^6 t^3\,\mathrm{d}t$ *using* RIGHT(4)

Solution to Problem 12

$$\begin{aligned}
\text{RIGHT}(4) &= \sum_{i=1}^{4} f\left(2 + i \cdot \tfrac{6-2}{4}\right) \cdot \tfrac{6-2}{4} \\
&= f(2 + 1 \cdot 1) \cdot 1 + f(2 + 2 \cdot 1) \cdot 1 \\
&\quad + f(2 + 3 \cdot 1) \cdot 1 + f(2 + 4 \cdot 1) \cdot 1 \\
&= f(3) + f(4) + f(5) + f(6) \\
&= 3^3 + 4^3 + 5^3 + 6^3 \\
&= \boxed{432}
\end{aligned}$$

Definition 13 (Monotonic functions)

Let f be a function on the interval I

- f is **increasing** on I if for all $a, b \in I$ if $a < b$ then $f(a) < f(b)$.
- f is **decreasing** on I if for all $a, b \in I$ if $a < b$ then $f(a) > f(b)$.
- f is **nondecreasing** on I if for all $a, b \in I$ if $a < b$ then $f(a) \leq f(b)$.
- f is **nonincreasing** on I if for all $a, b \in I$ if $a < b$ then $f(a) \geq f(b)$.

Theorem 14

If f is nondecreasing (or increasing) on $[a, b]$ then

$$\mathrm{LEFT}(n) \leq \int_a^b f(t)\,\mathrm{d}t \leq \mathrm{RIGHT}(n)$$

If f is nonincreasing (or decreasing) on $[a, b]$ then

$$\mathrm{LEFT}(n) \geq \int_a^b f(t)\,\mathrm{d}t \geq \mathrm{RIGHT}(n)$$

Combining Problems 11 and 12 and Theorem 14 we get

$$224 = \mathrm{LEFT}(4) \leq \int_2^6 t^3\,\mathrm{d}t \leq \mathrm{RIGHT}(4) = 432$$

(Note: Exact value is $\int_2^6 t^3\,\mathrm{d}t = 320$*)*

Problem 15

Suppose g is a nonincreasing function on the interval $[1, 11]$ and the following values are known

x	1	3	5	7	9	11
$g(x)$	20	15	14	14	6	2

1. *Bound* $\int_1^{11} g(x)\,\mathrm{d}x$ *using* LEFT(5) *and* RIGHT(5).
2. *Would these bounds still work if g were not assumed to be nonincreasing?*

Solution to Problem 15

1.
$$\mathrm{LEFT}(5) = 20\cdot 2 + 15\cdot 2 + 14\cdot 2 + 14\cdot 2 + 6\cdot 2 = 138$$
$$\mathrm{RIGHT}(5) = 15\cdot 2 + 14\cdot 2 + 14\cdot 2 + 6\cdot 2 + 2\cdot 2 = 102$$
$$\boxed{138 = \mathrm{LEFT}(5) \geq \int_1^{11} g(x)\,\mathrm{d}x \geq \mathrm{RIGHT}(5) = 102}$$

Solution to Problem 15

2. No

Let's look at LEFT(5) and RIGHT(5) for g from Problem 15 again

$$\text{LEFT}(5) = 20 \cdot 2 + 15 \cdot 2 + 14 \cdot 2 + 14 \cdot 2 + 6 \cdot 2 = 138$$

$$\text{RIGHT}(5) = 15 \cdot 2 + 14 \cdot 2 + 14 \cdot 2 + 6 \cdot 2 + 2 \cdot 2 = 102$$

Notice that most terms in LEFT(5) and RIGHT(5) agree.

So their difference is easy to compute

$$\text{RIGHT}(5) - \text{LEFT}(5) = 2 \cdot 2 - 20 \cdot 2$$

The next theorem makes this precise.

Theorem 16

If f is a function on $[a, b]$ then

$$\mathrm{RIGHT}(n) - \mathrm{LEFT}(n) = (f(b) - f(a)) \cdot \frac{b-a}{n}$$

Proof.

$$\begin{aligned}
\mathrm{RIGHT}(n) - \mathrm{LEFT}(n) &= \sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} \\
&\quad - \sum_{i=0}^{n-1} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} \\
&= f\left(a + n \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} + \sum_{i=1}^{n-1} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} \\
&\quad - f\left(a + 0 \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} - \sum_{i=1}^{n-1} f\left(a + i \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n}
\end{aligned}$$

Proof of Theorem 16 *(continued)*.

$$
\begin{aligned}
&= f\left(a + n \cdot \tfrac{b-a}{n}\right) \cdot \tfrac{b-a}{n} - f\left(a + 0 \cdot \tfrac{b-a}{n}\right) \cdot \tfrac{b-a}{n} \\
&= f\left(a + b - a\right) \cdot \tfrac{b-a}{n} - f\left(a\right) \cdot \tfrac{b-a}{n} \\
&= f\left(b\right) \cdot \tfrac{b-a}{n} - f\left(a\right) \cdot \tfrac{b-a}{n} \\
&= (f(b) - f(a)) \cdot \tfrac{b-a}{n}
\end{aligned}
$$

□

If f is monotonic on $[a, b]$ then we can use Theorem 16 to get some idea of how many intervals we need to estimate $\int_a^b f(t)\,\mathrm{d}t$ using RIGHT(n) and LEFT(n)

Problem 17

How many intervals are needed to estimate

$$\int_1^3 2^{(t^2)}\,\mathrm{d}t$$

within an error of 1 *using* RIGHT(n) *and* LEFT(n)*?*

Solution to Problem 17

Let $f(t) = 2^{(t^2)}$. Is f monotonic on $[1, 3]$? Yes

$$\frac{\mathrm{d}}{\mathrm{d}t} 2^{(t^2)} = 2t2^{(t^2)} \ln 2 > 0$$

So by Theorem 14

$$\text{LEFT}(n) \leq \int_1^3 2^{(t^2)}\,\mathrm{d}t \leq \text{RIGHT}(n)$$

Solution to Problem 17 *(continued)*

By Theorem 16

$$\begin{aligned}\text{RIGHT}(n) - \text{LEFT}(n) &= (f(b) - f(a)) \cdot \tfrac{b-a}{n} \\ &= (2^{(3^2)} - 2^{(1^2)}) \cdot \tfrac{3-1}{n} \\ &= (2^9 - 2^1) \cdot \tfrac{2}{n}\end{aligned}$$

In order to have $\text{RIGHT}(n) - \text{LEFT}(n) < 1$ we need to choose n so that

$$(2^9 - 2^1) \cdot \tfrac{2}{n} < 1$$

$$1020 = (2^9 - 2^1)(2) < n$$

RIGHT(1021) and LEFT(1021) will estimate $\int_1^3 2^{(t^2)}\,dt$ within 1