Math 152.02
Calculus with Analytic Geometry II

January 14, 2011

Integrals and area

Integrals and area

If $a < b$ then

$$\int_a^b f(x)\,\mathrm{d}x$$
$$= (\text{area under } f \text{ above } x\text{-axis}) - (\text{area above } f \text{ under } x\text{-axis})$$

Example 18

$$\int_{-3}^{0} \sqrt{9 - x^2}\,\mathrm{d}x = \frac{\pi \cdot 3^2}{4} = \boxed{\frac{9\pi}{4}}$$

Problem 19

$$\textit{Compute } \int_{-2}^{6} 2x - 5\,dx$$

by finding areas of regions between the graph of f and the x-axis.

Solution to Problem 19

x-intercept

$$0 = 2x - 5$$
$$x = \frac{5}{2}$$

(area under f above x-axis)

$$A_2 = \frac{1}{2} \cdot \left(6 - \frac{5}{2}\right)(2 \cdot 6 - 5)$$
$$= \frac{49}{4}$$

Solution to Problem 19 *(continued)*

(area above f under x-axis)

$$\begin{aligned} A_1 &= \tfrac{1}{2} \cdot \left(\tfrac{5}{2} - (-2)\right)\left(-2 \cdot (-2) + 5\right) \\ &= \tfrac{1}{2} \cdot \left(\tfrac{9}{2}\right)(9) \\ &= \tfrac{81}{4} \end{aligned}$$

$$\int_{-2}^{6} 2x - 5\,dx = A_2 - A_1 = \tfrac{49}{4} - \tfrac{81}{4} = \boxed{-8}$$

Example 20 (Area formulas seldom work)

$$\text{Evaluate } \int_{-5}^{3} \sqrt{25 - x^2}\,dx.$$

No area formula for this portion of a circle

Theorem 21 (Fundamental Theorem of Calculus I)

If f is continuous on the interval $[a, b]$ and $F'(t) = f(t)$ then

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a).$$

Example 22

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin 8x + x^4\right) = 8\cos 8x + 4x^3$ so by FTOC

$$\begin{aligned}\int_1^2 8\cos 8x + 4x^3\,\mathrm{d}x &= (\sin(8\cdot 2) + 2^4) - (\sin(8\cdot 1) + 1^4)\\ &= \sin 16 - \sin 8 + 15\end{aligned}$$

Notation

$$g(x)\Big|_{x=a}^{b} = g(b) - g(a)$$

or if x is clearly the variable to plug a and b into can write

$$g(x)\Big|_{a}^{b} = g(b) - g(a)$$

With this notation FTOC I (Fundamental Theorem of Calculus I) says
If f is continuous on the interval $[a, b]$ and $F'(t) = f(t)$ then

$$\int_a^b f(t)\,\mathrm{d}t = F(t)\Big|_{t=a}^{b}$$

FTOC says roughly

Computing area under f	$\approx$	Finding a function with derivative f

Suprising! Why should area and slopes be related?

Could prove FTOC from definition of integral (Definition 9) and definition of the derivative, but we won't.

Makes some sense visually

Interpreting integrals

Note on units for integrals

- If x measured in units u_1 and
- $f(x)$ measured in units u_2 then

$$\int_a^b f(x)\,\mathrm{d}x$$

has units $u_1 \cdot u_2$

Example 23

- If t is the time measured in hours
- $P(t)$ number of people working in a factory at time t.

$$\int_a^b P(t)\,\mathrm{d}t$$

is $\mathrm{people} \cdot \mathrm{hours}$ worked between time a and b

Integral of a rate of change

- If $F'(t)$ is the rate of change of some quantity $F(t)$
- then FTOC I say that

$$\int_a^b F'(t)\,\mathrm{d}t = F(b) - F(a)$$

This is **net change in** F from time a to b

Example 24

As in Example 4 from last week

- t time (in h)
- $v(t)$ velocity (rate of change of position) at time t (in $\mathrm{km/h}$)
- then

$$\int_0^3 v(t)\,\mathrm{d}t$$

is distance traveled (**net change in position**) in units $(\mathrm{km/h}) \cdot \mathrm{h} = \mathrm{km}$

Example 25

- t time (in years)
- $g(t)$ growth rate (in m/year) of person at age t
- then
$$\int_9^{16} g(t)\,\mathrm{d}t$$
is change in height (in m) of a person from age 9 to 16.

Average value of a function

Definition 26 (Average value of a function)

The **average value** of f on the interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a} \int_a^b f(t)\, \mathrm{d}t.$$

Where does this formula come from?

Let's derive it using

Philosophy of Calculus

- Estimate a quantity
- Figure out how to improve estimate
- Take limit

Average temperature

Let $f(t)$ be the temperature at time t.

Let's estimate the average temperature between 2am and 10pm.

- (First estimate with 2 intervals)
  - total time $= 10 - 2$
  - each interval is $\frac{10-2}{2}$
  - $$f_{\text{ave}} \approx \frac{f\left(2 + 1 \cdot \frac{10-2}{2}\right) + f\left(2 + 2 \cdot \frac{10-2}{2}\right)}{2}$$
- (Better estimate with 3 intervals)
  - total time $= 10 - 2$
  - each interval is $\frac{10-2}{3}$
  - $$f_{\text{ave}} \approx \frac{f\left(2 + 1 \cdot \frac{10-2}{3}\right) + f\left(2 + 2 \cdot \frac{10-2}{3}\right) + f\left(2 + 3 \cdot \frac{10-2}{3}\right)}{3}$$

Average temperature *(continued)*

- (Better estimate with n intervals)
  - total time $= 10 - 2$
  - each interval is $\frac{10-2}{n}$
  - $$\begin{aligned} f_{\text{ave}} &\approx \frac{f\left(2 + 1 \cdot \frac{10-2}{n}\right) + \cdots + f\left(2 + n \cdot \frac{10-2}{n}\right)}{n} \\ &= \sum_{i=1}^{n} f\left(2 + i \cdot \frac{10-2}{n}\right) \cdot \frac{1}{n} \\ &= \frac{1}{10-2} \sum_{i=1}^{n} f\left(2 + i \cdot \frac{10-2}{n}\right) \cdot \frac{10-2}{n} \end{aligned}$$

Average temperature *(continued)*

- (Take limit to get exact average)

$$\begin{aligned} f_{\text{ave}} &= \lim_{n\to\infty} \frac{1}{10-2} \sum_{i=1}^{n} f\left(2 + i \cdot \frac{10-2}{n}\right) \cdot \frac{10-2}{n} \\ &= \frac{1}{10-2} \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2 + i \cdot \frac{10-2}{n}\right) \cdot \frac{10-2}{n} \\ &= \frac{1}{10-2} \int_{2}^{10} f(t)\,\mathrm{d}t \end{aligned}$$

We've derived the formula in Definition 26.

Problem 27

Find the average value of the funtion

$$h(x) = \sqrt{4 - x^2}$$

on the interval $[-2, 2]$

Solution to Problem 27

$$\begin{aligned} h_{\text{ave}} &= \frac{1}{b-a} \int_a^b h(x)\,\mathrm{d}x \\ &= \frac{1}{2-(-2)} \int_{-2}^{2} \sqrt{4-x^2}\,\mathrm{d}x \\ &= \frac{1}{4} \cdot \frac{\pi \cdot 2^2}{2} \\ &= \boxed{\frac{\pi}{2}} \end{aligned}$$

Problem 28

- *t is time measured in days since Jan. 1, 2003*
- *$R(t)$ is the distance from the earth to the sun at time t*

What does

$$\frac{1}{365-0}\int_0^{365} R(t)\,dt$$

represent?

Solution to Problem 28

Average distance from earth to the sun in 2003

Properties of definite integrals

Theorem 29

If f is integrable then

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

Theorem 30

If f is integrable then

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$$

Theorem 31

If f is integrable then

$$\int_a^b cf(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

Proof.

$$\begin{aligned}\int_a^b cf(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^{n} cf(a+i\Delta x)\Delta x\\ &= \lim_{n\to\infty} c\sum_{i=1}^{n} f(a+i\Delta x)\Delta x\\ &= c\lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x\\ &= c\int_a^b f(x)\,\mathrm{d}x\end{aligned}$$

Theorem 32

If f and g are integrable then

$$\int_a^b f(x) + g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

Proof.

$$\begin{aligned}\int_a^b f(x) + g(x)\,\mathrm{d}x &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(a+i\Delta x) + g(a+i\Delta x)\right)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(a+i\Delta x)\Delta x + g(a+i\Delta x)\Delta x\right)\end{aligned}$$

Proof of Theorem 32 *(continued)*.

$$
\begin{aligned}
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(f(a+i\Delta x)\Delta x + g(a+i\Delta x)\Delta x\right) \\
&= \lim_{n\to\infty} \left(\sum_{i=1}^{n} f(a+i\Delta x)\Delta x + \sum_{i=1}^{n} g(a+i\Delta x)\Delta x \right) \\
&= \left(\lim_{n\to\infty} \sum_{i=1}^{n} f(a+i\Delta x)\Delta x \right) + \left(\lim_{n\to\infty} \sum_{i=1}^{n} g(a+i\Delta x)\Delta x \right) \\
&= \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x
\end{aligned}
$$

□

Problem 33

Compute $\int_7^{16} t(v) - 7u(v)\,\mathrm{d}v$ *given*

- $\int_7^{16} u(v)\,\mathrm{d}v = -2$
- $\int_7^{16} t(v)\,\mathrm{d}v = 12$

Solution to Problem 33

$$\begin{aligned}\int_7^{16} t(v) - 7u(v)\,\mathrm{d}v &= \int_7^{16} t(v)\,\mathrm{d}v - \int_7^{16} 7u(v)\,\mathrm{d}v \\ &= \int_7^{16} t(v)\,\mathrm{d}v - 7\int_7^{16} u(v)\,\mathrm{d}v \\ &= 12 - 7(-2) \\ &= \boxed{26}\end{aligned}$$

Warning

$$\int_a^b f(x) \cdot g(x)\,\mathrm{d}x \neq \int_a^b f(x)\,\mathrm{d}x \cdot \int_a^b g(x)\,\mathrm{d}x$$

$$\int_a^b \frac{f(x)}{g(x)}\,\mathrm{d}x \neq \frac{\int_a^b f(x)\,\mathrm{d}x}{\int_a^b g(x)\,\mathrm{d}x}$$

Example 34

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & 1 < x \leq 2 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & 0 \leq x \leq 1 \\ 1, & 1 < x \leq 2 \end{cases}$$

$$\int_0^2 f(x)g(x)\,\mathrm{d}x = \int_0^2 0\,\mathrm{d}x = 0$$

$$\text{but} \quad \int_0^2 f(x)\,\mathrm{d}x \cdot \int_0^2 g(x)\,\mathrm{d}x = 1 \cdot 1 = 1$$

Definition 35

- f is **even** if $f(-x) = f(x)$
- f is **odd** if $f(-x) = -f(x)$

Note: Most functions are neither. One function is both.

Theorem 36

If f is even then

$$\int_0^a f(x)\,dx = \int_{-a}^0 f(x)\,dx = \frac{1}{2}\int_{-a}^a f(x)\,dx$$

If f is odd then

$$\int_0^a f(x)\,dx = -\int_{-a}^0 f(x)\,dx$$

and

$$\int_{-a}^a f(x)\,dx = 0$$

Problem 37

Suppose

- $\int_2^4 Q(s)\,\mathrm{d}s = 3$
- $\int_4^{20} Q(s)\,\mathrm{d}s = -9$

Evaluate

$$\int_2^{20} Q(s)\,\mathrm{d}s$$

Solution to Problem 37

$$\begin{aligned}\int_2^{20} Q(s)\,\mathrm{d}s &= \int_2^{4} Q(s)\,\mathrm{d}s + \int_4^{20} Q(s)\,\mathrm{d}s \\ &= 3 + (-9) \\ &= \boxed{-6}\end{aligned}$$

Problem 38

Compute $\int_2^8 y(u)\,\mathrm{d}u$ *given*

- *y is an odd function*
- $\int_{-2}^8 y(u)\,\mathrm{d}u = 17$

Solution to Problem 38

$$\begin{aligned}\int_2^8 y(u)\,\mathrm{d}u &= \int_{-2}^8 y(u)\,\mathrm{d}u - \int_{-2}^2 y(u)\,\mathrm{d}u \\ &= 17 - 0 \\ &= \boxed{17}\end{aligned}$$

Problem 39

Compute $\int_3^7 h(r)\,\mathrm{d}r$ *given*

- *h is an even function*
- $\int_0^7 h(r)\,\mathrm{d}r = 4$
- $\int_{-3}^7 h(r)\,\mathrm{d}r = -10$

Solution to Problem 39

$$\begin{aligned}\int_3^7 h(r)\,\mathrm{d}r &= \int_{-3}^7 h(r)\,\mathrm{d}r - \int_{-3}^3 h(r)\,\mathrm{d}r\\ &= \int_{-3}^7 h(r)\,\mathrm{d}r - 2\int_{-3}^0 h(r)\,\mathrm{d}r\\ &= \int_{-3}^7 h(r)\,\mathrm{d}r - 2\left(\int_{-3}^7 h(r)\,\mathrm{d}r - \int_0^7 h(r)\,\mathrm{d}r\right)\\ &= -\int_{-3}^7 h(r)\,\mathrm{d}r + 2\int_0^7 h(r)\,\mathrm{d}r\\ &= -(-10) + 2(4) = \boxed{18}\end{aligned}$$

Bounding on integrals

Theorem 40

If $m \leq f(x) \leq M$ *for* $x \in [a, b]$ *then*

$$m(b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M(b-a)$$

Theorem 41

If $f(x) \leq g(x)$ *for* $x \in [a, b]$ *then*

$$\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$$

Problem 42

Show that

$$-199 \leq \int_1^{200} \cos(x) \cdot \sin(x^3)\,\mathrm{d}x \leq 199$$

Solution to Problem 42

For all $x \in [1, 200]$

$$|\cos(x) \cdot \sin(x^3)| = |\cos(x)| \cdot |\sin(x^3)| \leq 1$$

so

$$-1 \leq \cos(x) \cdot \sin(x^3) \leq 1$$

By Theorem 40

$$-1 \cdot (200 - 1) \leq \int_1^{200} \cos(x) \cdot \sin(x^3)\,\mathrm{d}x \leq 1 \cdot (200 - 1)$$

$$-199 \leq \int_1^{200} \cos(x) \cdot \sin(x^3)\,\mathrm{d}x \leq 199$$

Problem 43

For each pairs of integrals decide which is the larger

1. $\int_0^{\frac{\pi}{4}} \cos(x)\,\mathrm{d}x$ *and* $\int_0^{\frac{\pi}{4}} \sin(x)\,\mathrm{d}x$
2. $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(x)\,\mathrm{d}x$ *and* $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)\,\mathrm{d}x$

Solution to Problem 43

1. For all $x \in [0, \frac{\pi}{4}]$
$$\cos(x) \geq \sin(x)$$
so
$$\int_0^{\frac{\pi}{4}} \cos(x)\,\mathrm{d}x \geq \int_0^{\frac{\pi}{4}} \sin(x)\,\mathrm{d}x$$

2. For all $x \in [\frac{\pi}{4}, \frac{\pi}{2}]$
$$\cos(x) \leq \sin(x)$$
so
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(x)\,\mathrm{d}x \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)\,\mathrm{d}x$$

Antiderivatives

Definition 44 (An antiderivative)

$F(x)$ is **an antiderivative** of $f(x)$ if $F'(x) = f(x)$.

Example 45

$$\frac{d}{dx}\left(x^2\cos(4x+3)\right) = 2x\cos(4x+3) + 4x^2\sin(4x+3)$$

so

$$x^2\cos(4x+3)$$

is an antiderivative of

$$2x\cos(4x+3) + 4x^2\sin(4x+3)$$

$x^2\cos(4x+3) + 30$ is another antiderivative.

Motivation

Why do we care about finding antiderivatives?

FTOC I says that computing

$$\int_a^b f(t)\,\mathrm{d}t$$

is easy if we have an antiderivative F of f.

Example 46

From Example 45 above

$$\begin{aligned}\int_1^7 2x\cos(4x+3)+4x^2\sin(4x+3)\,\mathrm{d}x \\ &= x^2\cos(4x+3)+30\Big|_1^7 \\ &= (7^2\cos(4\cdot 7+3)+30)-(1^2\cos(4\cdot 1+3)+30) \\ &= \boxed{49\cos(31)-\cos(7)}\end{aligned}$$

Notice in Example 45 we could have added any constant to $x^2 \cos(4x+3)$ and we would have had another antiderivative of $2x\cos(4x+3) + 4x^2\sin(4x+3)$

We usually add an unspecified constant to remind us that there are many antiderivatives.

Definition 47 (The antiderivative)

The antiderivative of $f(x)$ is the set of all antiderivatives of $f(x)$.

Theorem 48

If f is **continuous** *and $F'(x) = f(x)$ then every antiderivative of f is of the form*

$$F(x) + C$$

for some constant C.

What if f is **not** continuous?

The antiderivative of a noncontinuous function

Let

$$F(x) = \begin{cases} \ln x + 14, & x > 0 \\ \ln(-x) + 8, & x < 0 \end{cases}$$

Then

$$F'(x) = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{-1}{-x}, & x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{x}, & x < 0 \end{cases} = \frac{1}{x}$$

So $F(x)$ is an antiderivative of $\frac{1}{x}$.

Any choice of constants (14 and 8 weren't special) gives same result.

Thus **the antiderivative** of $\frac{1}{x}$ is

$$F(x) = \begin{cases} \ln x + C_1, & x > 0 \\ \ln(-x) + C_2, & x < 0 \end{cases} = \begin{cases} \ln|x| + C_1, & x > 0 \\ \ln|x| + C_2, & x < 0 \end{cases}$$

On the other hand

- Main reason we care about antiderivatives is the FTOC.
- FTOC only applies if f is integrable on $[a, b]$
- $\frac{1}{x}$ is not integrable on intervals containing 0
- so in applications we only use one of the two constants at a time

Example 49

$$\int_{-3}^{-2} \frac{1}{x}\,dx = \ln|x| + C_2\Big|_{-3}^{-2} = (\ln|-2| + C_2) - (\ln|-3| + C_2) = \ln 2 - \ln 3$$

Example 50

$$\int_{-4}^{7} \frac{1}{x}\,dx \quad \text{cannot be evaluated using FTOC}$$

Notational warning

By convention we say that $F(x) + C$ is **the antiderivative** of $f(x)$ whenever $F'(x) = f(x)$ even when this is technically incorrect.

Notation

$$\int f(x)\,\mathrm{d}x = F(x) + C$$

means $F(x) + C$ is the antiderivative of $f(x)$

Terminology

Since FTOC links antidifferentiation and integration we also call antiderivatives (indefinite) integrals.

The following statements all mean the same thing:

- $f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x)$
- $\int f(x)\,\mathrm{d}x = F(x) + C$
- $f(x)$ is the derivative of $F(x)$
- $F(x) + C$ is the **antiderivative** of $f(x)$
- $F(x) + C$ is the **indefinite integral** of $f(x)$
- $F(x) + C$ is the **integral** of $f(x)$

Problem 51

Check the following integrals

1. $\int (6x + 3e^x) \cdot \cos(3x^2 + 3e^x) \, dx = \sin(3x^2 + 3e^x) + C$
2. $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Solution to Problem 51

1. $$\frac{d}{dx} \sin(3x^2 + 3e^x) = (6x + 3e^x) \cdot \cos(3x^2 + 3e^x) \checkmark$$

2. $$\begin{aligned} \frac{d}{dx} \ln|\sec x + \tan x| &= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ &= (\sec x) \cdot \frac{\tan x + \sec x}{\sec x + \tan x} \\ &= \sec x \checkmark \end{aligned}$$

Note: Similar to $\frac{1}{x}$ antiderivative of $\sec x$ should have a different C for each interval $[\frac{(2n-1)\pi}{2}, \frac{(2n+1)\pi}{2}]$ but nobody does this.

(See backboard)

Basic integration rules

Each rule for differentiation gives us a rule for integration

From

$$c\,\tfrac{\mathrm{d}}{\mathrm{d}x}F(x) = \tfrac{\mathrm{d}}{\mathrm{d}x}\Big(cF(x)\Big)$$

we get

Theorem 52 (Constant rule for integration)

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x$$

Proof of Theorem 52.

Suppose $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

We have the derivative rule

$$c\frac{\mathrm{d}}{\mathrm{d}x}F(x) = \frac{\mathrm{d}}{\mathrm{d}x}\big(cF(x)\big)$$

Reinterpreting this rule as an antiderivative gives

$$\int c\frac{\mathrm{d}}{\mathrm{d}x}F(x)\,\mathrm{d}x = cF(x) + C.$$

Thus we may conclude

$$\begin{aligned}\int cf(x)\,\mathrm{d}x &= \int c\frac{\mathrm{d}}{\mathrm{d}x}F(x)\,\mathrm{d}x \\ &= cF(x) + C \\ &= c(F(x) + C_2) \\ &= c\int f(x)\,\mathrm{d}x.\end{aligned}$$

□

From

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) + \frac{\mathrm{d}}{\mathrm{d}x}G(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(F(x) + G(x)\Big)$$

we get

Theorem 53 (Sum rule for integration)

$$\int f(x) + g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

Proof of Theorem 53.

Suppose $\frac{d}{dx}F(x) = f(x)$ and $\frac{d}{dx}G(x) = g(x)$.

We have the derivative rule

$$\frac{d}{dx}F(x) + \frac{d}{dx}G(x) = \frac{d}{dx}\Big(F(x) + G(x)\Big)$$

Reinterpreting this rule as an antiderivative gives

$$\int \frac{d}{dx}F(x) + \frac{d}{dx}G(x)\,dx = F(x) + G(x) + C.$$

Thus we may conclude

$$\begin{aligned}\int f(x) + g(x)\,dx &= \int \frac{d}{dx}F(x) + \frac{d}{dx}G(x)\,dx \\ &= F(x) + G(x) + C \\ &= \int f(x)\,dx + \int g(x)\,dx\end{aligned}$$

Note: We drop constants when we have integrals on both sides of an equation. □

Basic Integrals

Each basic derivative gives us a basic integral

Table 1: Basic integrals to memorize

differentiation rule	**integration rule**
$\frac{\mathrm{d}}{\mathrm{d}x}x^{r+1} = (r+1)x^r$	$\int x^r \,\mathrm{d}x = \frac{1}{r+1}x^{r+1} + C$ if $r \neq -1$
$\frac{\mathrm{d}}{\mathrm{d}x} \ln\lvert x\rvert = \frac{1}{x}$	$\int \frac{1}{x}\,\mathrm{d}x = \ln\lvert x\rvert + C$
$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$	$\int \sin x \,\mathrm{d}x = -\cos x + C$
$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$	$\int \cos x \,\mathrm{d}x = \sin x + C$
$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$	$\int e^x \,\mathrm{d}x = e^x + C$
$\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\,\mathrm{d}x = \arctan x + C$
$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + C$

More basic integrals

You also know a few more derivative rules

Table 2: More basic integrals to memorize

differentiation rule	**integration rule**
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}a^x = (\ln a)a^x$	$\int a^x\,dx = \frac{a^x}{\ln a} + C$

Techniques of integration

Advanced derivative rules give us techniques of integration

differentiation rule	**technique of integration**
chain rule	u-substitution (§7.1)
product rule	integration by parts (§7.2)

We will return to these integration techniques later.

Problem 54

Find a formula for $\int 10e^x + 7\sin x\,dx$

Solution to Problem 54

$$\begin{aligned}\int 10e^x + 7\sin x\,dx &= \int 10e^x\,dx + \int 7\sin x\,dx && \text{(Sum rule)}\\ &= 10\int e^x\,dx + 7\int \sin x\,dx && \text{(Constant rule)}\\ &= \boxed{10e^x - 7\cos x + C} && \text{(Table 1)}\end{aligned}$$

Check your answer!

$$\frac{d}{dx}(10e^x - 7\cos x) = 10e^x + 7\sin x \checkmark$$

Problem 55

Find a formula for $\int \frac{18}{\sqrt{1-t^2}} - 8t^{22}\,dt$

Solution to Problem 55

$$\begin{aligned}\int \frac{18}{\sqrt{1-t^2}} - 8t^{22}\,dt &= \int 18\cdot\frac{1}{\sqrt{1-t^2}}\,dt - \int 8\cdot t^{22}\,dt \quad \text{(Sum rule)}\\ &= 18\int \frac{1}{\sqrt{1-t^2}}\,dt - 8\int t^{22}\,dt \quad \text{(Const. rule)}\\ &= \boxed{18\arcsin t - 8\cdot\frac{t^{23}}{23} + C} \qquad \text{(Table 1)}\end{aligned}$$

Check your answer!

$$\frac{d}{dt}\left(18\arcsin t - \frac{8}{23}\cdot t^{23}\right) = \frac{18}{\sqrt{1-t^2}} - \frac{8}{23}\cdot 23t^{22} = \frac{18}{\sqrt{1-t^2}} - 8t^{22}\ \checkmark$$

Problem 56

Find a formula for $\int \frac{\arcsin(3-\pi^2)}{\sqrt{2}} + \sec^2 u \, du$

Solution to Problem 56

$$\int \frac{\arcsin(3-\pi^2)}{\sqrt{2}} + \cos u \, du$$

$$= \int \frac{\arcsin(3-\pi^2)}{\sqrt{2}} \, du + \int \sec^2 u \, du \quad \text{(Sum rule)}$$

$$= \boxed{\frac{\arcsin(3-\pi^2)}{\sqrt{2}} \cdot u + \tan u + C} \qquad \text{(Tables 1 and 2)}$$

Check your answer!

$$\frac{d}{dt}\left(\frac{\arcsin(3-\pi^2)}{\sqrt{2}} \cdot u + \tan u\right) = \frac{\arcsin(3-\pi^2)}{\sqrt{2}} + \sec^2 u \ \checkmark$$

Problem 57

Compute $\int_1^2 \frac{3\sqrt{y}(1-6\sqrt[6]{y})^2}{\sqrt[3]{y}}\,\mathrm{d}y$

Solution to Problem 57

$$\begin{aligned}
\int_1^2 \frac{3\sqrt{y}(1-6\sqrt[6]{y})^2}{\sqrt[3]{y}}\,\mathrm{d}y &= \int_1^2 \frac{3y^{\frac{1}{2}}(1-12y^{\frac{1}{6}}+36y^{\frac{2}{6}})}{y^{\frac{1}{3}}}\,\mathrm{d}y \\
&= \int_1^2 \frac{3y^{\frac{1}{2}}-36y^{\frac{4}{6}}+108y^{\frac{5}{6}})}{y^{\frac{1}{3}}}\,\mathrm{d}y \\
&= \int_1^2 3y^{\frac{1}{6}}-36y^{\frac{2}{6}}+108y^{\frac{3}{6}}\,\mathrm{d}y \\
&= \int_1^2 3y^{\frac{1}{6}}-36y^{\frac{1}{3}}+108y^{\frac{1}{2}}\,\mathrm{d}y \\
&= 3\cdot\frac{6}{7}y^{\frac{7}{6}}-36\cdot\frac{3}{4}y^{\frac{4}{3}}+108\cdot\frac{2}{3}y^{\frac{3}{2}}\Big|_1^2 \\
&= \frac{18}{7}y^{\frac{7}{6}}-27y^{\frac{4}{3}}+72y^{\frac{3}{2}}\Big|_1^2 \\
&= \boxed{\left(\frac{18}{7}\cdot 2^{\frac{7}{6}}-27\cdot 2^{\frac{4}{3}}+72\cdot 2^{\frac{3}{2}}\right)-\left(\frac{18}{7}+45\right)}
\end{aligned}$$

Problem 58

Find an antiderivative $G(x)$ of $g(x) = \sin x + 7$ satisfying $G(\pi) = -20$.

Solution to Problem 58

$$\begin{aligned} G(x) &= \int \sin x + 7 \, dx \\ &= -\cos x + 7x + C \end{aligned}$$

Use fact that $G(\pi) = -20$ to solve for C.

$$-20 = G(\pi) = -\cos \pi + C$$

So

$$C = -20 + \cos \pi = -20 + (-1) = -21$$

$$\boxed{G(x) = -\cos x + 7x - 21}$$

Problem 59

The average value of $h(x) = x^3 - 3x^2$ *on* $[-a, a]$ *is* 8 *solve for* a.

Solution to Problem 59

$$\begin{aligned} h_{\text{ave}} &= \frac{1}{a - (-a)} \int_{-a}^{a} x^3 - 3x^2 \, dx \\ &= \frac{1}{2a} \cdot \left(\tfrac{1}{4}x^4 - x^3\right)\Big|_{-a}^{a} \\ &= \frac{1}{2a} \cdot \left(\left[\tfrac{1}{4}(-a)^4 - (-a)^3\right] - \left(\tfrac{1}{4}a^4 - a^3\right)\right) \\ &= \frac{1}{2a} \cdot \left(\tfrac{1}{4}a^4 + a^3 - \tfrac{1}{4}a^4 + a^3\right) \\ &= \frac{1}{2a} \cdot (2a^3) \\ &= a^2 \end{aligned}$$

Use fact that $h_{\text{ave}} = 8$ to solve for a.

$$8 = h_{\text{ave}} = a^2 \quad \text{so} \quad \boxed{a = \pm\sqrt{8}}$$