Math 152.02
Calculus with Analytic Geometry II

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Basic integration rules

Each rule for differentiation gives us a rule for integration

From

$$c\,\tfrac{\mathrm{d}}{\mathrm{d}x}F(x) = \tfrac{\mathrm{d}}{\mathrm{d}x}\Big(cF(x)\Big)$$

we get

Theorem 52 (Constant rule for integration)

$$\int cf(x)\,\mathrm{d}x = c\int f(x)\,\mathrm{d}x$$

Proof of Theorem 52.

Suppose $\frac{d}{dx}F(x) = f(x)$.

We have the derivative rule

$$c\frac{d}{dx}F(x) = \frac{d}{dx}\big(cF(x)\big)$$

Reinterpreting this rule as an antiderivative gives

$$\int c\frac{d}{dx}F(x)\,dx = cF(x) + C.$$

Thus we may conclude

$$\begin{aligned}\int cf(x)\,dx &= \int c\frac{d}{dx}F(x)\,dx \\ &= cF(x) + C \\ &= c(F(x) + C_2) \\ &= c\int f(x)\,dx.\end{aligned}$$

□

From

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) + \frac{\mathrm{d}}{\mathrm{d}x}G(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(F(x) + G(x)\Big)$$

we get

Theorem 53 (Sum rule for integration)

$$\int f(x) + g(x)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

Proof of Theorem 53.

Suppose $\frac{d}{dx}F(x) = f(x)$ and $\frac{d}{dx}G(x) = g(x)$.

We have the derivative rule

$$\frac{d}{dx}F(x) + \frac{d}{dx}G(x) = \frac{d}{dx}\Big(F(x) + G(x)\Big)$$

Reinterpreting this rule as an antiderivative gives

$$\int \frac{d}{dx}F(x) + \frac{d}{dx}G(x)\,dx = F(x) + G(x) + C.$$

Thus we may conclude

$$\begin{aligned}\int f(x) + g(x)\,dx &= \int \frac{d}{dx}F(x) + \frac{d}{dx}G(x)\,dx \\ &= F(x) + G(x) + C \\ &= \int f(x)\,dx + \int g(x)\,dx\end{aligned}$$

Note: We drop constants when we have integrals on both sides of an equation. □

Basic Integrals

Each basic derivative gives us a basic integral

Table 1: Basic integrals to memorize

differentiation rule	**integration rule**
$\frac{d}{dx}x^{r+1} = (r+1)x^r$	$\int x^r \, dx = \frac{1}{r+1}x^{r+1} + C$ if $r \neq -1$
$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$	$\int \frac{1}{x}\, dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x \, dx = -\cos x + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x \, dx = \sin x + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x \, dx = e^x + C$
$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\, dx = \arctan x + C$
$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin x + C$

More basic integrals

You also know a few more derivative rules

Table 2: More basic integrals to memorize

differentiation rule	integration rule
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$
$\frac{d}{dx}a^x = (\ln a)a^x$	$\int a^x\,dx = \frac{a^x}{\ln a} + C$

Techniques of integration

Advanced derivative rules give us techniques of integration

differentiation rule	**technique of integration**
chain rule	u-substitution (§7.1)
product rule	integration by parts (§7.2)

We will return to these integration techniques later.

Problem 54

Find a formula for $\int 10e^x + 7\sin x \, dx$

Solution to Problem 54

$$\begin{aligned} \int 10e^x + 7\sin x \, dx &= \int 10e^x \, dx + \int 7\sin x \, dx && \text{(Sum rule)} \\ &= 10\int e^x \, dx + 7\int \sin x \, dx && \text{(Constant rule)} \\ &= \boxed{10e^x - 7\cos x + C} && \text{(Table 1)} \end{aligned}$$

Check your answer!

$$\frac{d}{dx}(10e^x - 7\cos x) = 10e^x + 7\sin x \checkmark$$

Problem 55

Find a formula for $\int \frac{18}{\sqrt{1-t^2}} - 8t^{22}\,\mathrm{d}t$

Solution to Problem 55

$$\begin{aligned}\int \frac{18}{\sqrt{1-t^2}} - 8t^{22}\,\mathrm{d}t &= \int 18 \cdot \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t - \int 8 \cdot t^{22}\,\mathrm{d}t \quad \text{(Sum rule)} \\ &= 18 \int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t - 8 \int t^{22}\,\mathrm{d}t \quad \text{(Const. rule)} \\ &= \boxed{18 \arcsin t - 8 \cdot \tfrac{t^{23}}{23} + C} \qquad \text{(Table 1)}\end{aligned}$$

Check your answer!

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(18 \arcsin t - \frac{8}{23} \cdot t^{23}\right) = \frac{18}{\sqrt{1-t^2}} - \frac{8}{23} \cdot 23t^{22} = \frac{18}{\sqrt{1-t^2}} - 8t^{22} \checkmark$$

Problem 56

Find a formula for $\int \frac{\arcsin(3-\pi^2)}{\sqrt{2}} + \sec^2 u \, \mathrm{d}u$

Solution to Problem 56

$$\int \frac{\arcsin(3-\pi^2)}{\sqrt{2}} + \cos u \, \mathrm{d}u$$

$$= \int \frac{\arcsin(3-\pi^2)}{\sqrt{2}} \, \mathrm{d}u + \int \sec^2 u \, \mathrm{d}u \quad \text{(Sum rule)}$$

$$= \boxed{\frac{\arcsin(3-\pi^2)}{\sqrt{2}} \cdot u + \tan u + C} \qquad \text{(Tables 1 and 2)}$$

Check your answer!

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\arcsin(3-\pi^2)}{\sqrt{2}} \cdot u + \tan u\right) = \frac{\arcsin(3-\pi^2)}{\sqrt{2}} + \sec^2 u \ \checkmark$$

Problem 57

Compute $\int_1^2 \frac{3\sqrt{y}(1-6\sqrt[6]{y})^2}{\sqrt[3]{y}}\,\mathrm{d}y$

Solution to Problem 57

$$\begin{aligned}
\int_1^2 \frac{3\sqrt{y}(1-6\sqrt[6]{y})^2}{\sqrt[3]{y}}\,\mathrm{d}y &= \int_1^2 \frac{3y^{\frac{1}{2}}(1-12y^{\frac{1}{6}}+36y^{\frac{2}{6}})}{y^{\frac{1}{3}}}\,\mathrm{d}y \\
&= \int_1^2 \frac{3y^{\frac{1}{2}}-36y^{\frac{4}{6}}+108y^{\frac{5}{6}})}{y^{\frac{1}{3}}}\,\mathrm{d}y \\
&= \int_1^2 3y^{\frac{1}{6}}-36y^{\frac{2}{6}}+108y^{\frac{3}{6}}\,\mathrm{d}y \\
&= \int_1^2 3y^{\frac{1}{6}}-36y^{\frac{1}{3}}+108y^{\frac{1}{2}}\,\mathrm{d}y \\
&= 3\cdot\frac{6}{7}y^{\frac{7}{6}}-36\cdot\frac{3}{4}y^{\frac{4}{3}}+108\cdot\frac{2}{3}y^{\frac{3}{2}}\Big|_1^2 \\
&= \frac{18}{7}y^{\frac{7}{6}}-27y^{\frac{4}{3}}+72y^{\frac{3}{2}}\Big|_1^2 \\
&= \boxed{\left(\frac{18}{7}\cdot 2^{\frac{7}{6}}-27\cdot 2^{\frac{4}{3}}+72\cdot 2^{\frac{3}{2}}\right)-\left(\frac{18}{7}+45\right)}
\end{aligned}$$

Problem 58

Find an antiderivative $G(x)$ *of* $g(x) = \sin x + 7$ *satisfying* $G(\pi) = -20$.

Solution to Problem 58

$$\begin{aligned} G(x) &= \int \sin x + 7 \, \mathrm{d}x \\ &= -\cos x + 7x + C \end{aligned}$$

Use fact that $G(\pi) = -20$ to solve for C.

$$-20 = G(\pi) = -\cos \pi + 7\pi + C$$

So

$$C = -20 + \cos \pi - 7\pi = -20 + (-1) - 7\pi = -21 - 7\pi$$

$$\boxed{G(x) = -\cos x + 7x - 21 - 7\pi}$$

Problem 59

The average value of $h(x) = x^3 - 3x^2$ *on* $[-a, a]$ *is* -8 *solve for* a.

Solution to Problem 59

$$\begin{aligned} h_{\text{ave}} &= \frac{1}{a-(-a)} \int_{-a}^{a} x^3 - 3x^2 \, dx \\ &= \frac{1}{2a} \cdot \left(\tfrac{1}{4}x^4 - x^3\right)\Big|_{-a}^{a} \\ &= \frac{1}{2a} \cdot \Big([\tfrac{1}{4}a^4 - a^3] - (\tfrac{1}{4}(-a)^4 - (-a)^3) \Big) \\ &= \frac{1}{2a} \cdot \left(\tfrac{1}{4}a^4 - a^3 - \tfrac{1}{4}a^4 + a^3\right) \\ &= \frac{1}{2a} \cdot \left(-2a^3\right) \\ &= -a^2 \end{aligned}$$

Use fact that $h_{\text{ave}} = -8$ to solve for a.

$$-8 = h_{\text{ave}} = -a^2 \quad \text{so} \quad \boxed{a = \pm\sqrt{8}}$$

Differential equations

A standard equation is an equation satisfied by a number

Example 60

$x = 2$ is solution to the equation

$$x^3 - 4x^2 + 7x + 1 = 7$$

since

$$2^3 - 4 \cdot 2^2 + 7 \cdot 2 + 1 = 8 - 16 + 14 + 1 = 7$$

A **differential equation** is an equation satisfied by a **function**.

Solving differential equation may be difficult but checking the solution is easy.

Problem 61

Show that $y = e^{7x}$ *is a solution to the differential equation*

$$y' = 7y$$

Solution to Problem 61

If $y = e^{7x}$ then

$$y' = 7e^{7x}$$

and

$$7y = 7e^{7x}$$

so

$$y' = 7e^{7x} = 7y$$

Problem 62

Show that $y = x^3 + 3x - 2$ *is a solution to the differential equation*

$$6xy - 6y' = x^3y'' - 12x - 18$$

Solution to Problem 62

If $y = x^3 + 3x - 2$ then

$$y' = 3x^2 + 3$$

and

$$y'' = 6x$$

so

$$\begin{aligned} 6xy - 6y' &= 6x(x^3 + 3x - 2) - 6(3x^2 + 3) \\ &= 6x^4 + 18x^2 - 12x - 18x^2 - 18 \\ &= 6x^4 - 12x - 18 \end{aligned}$$

and

$$\begin{aligned} x^3y'' - 12x - 18 &= x^3(6x) - 12x - 18 \\ &= 6x^4 - 12x - 18 \end{aligned}$$

Terminology

Differential equations usually have multiple solutions. The set of all solutions is the **general solution** of the differential equation.

Further constraints on solutions to different equations called **initial conditions** are sometimes imposed

Example 63

The general solution to $y' = 5y$ is $y = Ae^{5t}$ (we will prove this using substitution)

If we further impose the intial condition $y(0) = 20$ then the unique solution is $y = 20e^{5t}$

Integration allows us to solve simple differential equations of the form

$$y' = f(x)$$

Problem 64

Find the general solution to the differential equation

$$y' = \csc^2 t - 20 \cdot 6^t + 12$$

Solution to Problem 64

$$y = \int \csc^2 t - 20 \cdot 6^t + 12 \, \mathrm{d}t$$
$$= \boxed{-\cot t - \frac{20}{\ln 6} \cdot 6^t + 12t + C}$$

Check your answer!

$$y' = \frac{\mathrm{d}}{\mathrm{d}t}\left(-\cot t - \frac{20}{\ln 6} \cdot 6^t + 12t + C\right)$$

$$= -(-\csc^2 t) - \frac{20}{\ln 6} \cdot \ln 6 \cdot 6^t + 12$$

$$= \csc^2 t - 20 \cdot 6^t + 12 \checkmark$$

Problem 65

Find the solution to the differential equation

$$y'' = -8$$

satisfying the initial conditions

$$y(0) = 200, \qquad y'(0) = 10$$

Solution to Problem 65

$$y' = \int -8 \, \mathrm{d}t$$
$$= -8t + C_1$$

$$10 = y'(0) = -8 \cdot 0 + C_1$$

so $C_1 = 10$

$$y' = -8t + 10$$

$$y = \int -8t + 10 \, \mathrm{d}t$$
$$= -4t^2 + 10t + C_2$$

$$200 = -4 \cdot 0^2 + 10 \cdot 0 + C_2$$

so $C_2 = 200$

$$\boxed{y = -4t^2 + 10t + 200}$$

Motion

Definition 66

If the **position** of an object at time t is given by the function

$$s(t)$$

then its **velocity** is

$$v(t) = \frac{ds}{dt}$$

and its **acceleration** is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Problem 67

An object is dropped from rest at an initial height of $17\mathrm{m}$ *with constant acceration of* $-9.81\mathrm{m}/\mathrm{s}$*. Give a differential equation and initial conditions satisfied by the position function.*

Solution to Problem 68

Constant acceleration of $-9.81\mathrm{m}/\mathrm{s}^2$ gives us the differential equation

$$s'' = -9.81$$

Initial height of $17\mathrm{m}$ gives the initial condition

$$s(0) = 17$$

The object is dropped from rest giving the initial condition

$$s'(0) = 0$$

$$\boxed{s'' = -9.81, \quad s'(0) = 0, \quad s(0) = 17}$$

Problem 68

An object is dropped from rest on the moon from a height of $2\mathrm{m}$ *and takes* $1.57\mathrm{s}$ *to hit the ground. What is the (constant) acceleration of gravity at the moon's surface?*

Solution to Problem 68

We have the following diff. eq. and initial conditions

$$s'' = a, \qquad s'(0) = 0,$$
$$s(0) = 2, \qquad s(1.57) = 0$$

$$s' = \int a\,\mathrm{d}t$$
$$= at + C_1$$

$$0 = s'(0) = a \cdot 0 + C_1$$

$$\text{so } C_1 = 0$$

$$s = \int at\,\mathrm{d}t$$
$$= \tfrac{at^2}{2} + C_2$$

$$2 = \tfrac{a \cdot 0^2}{2} + C_2$$

$$\text{so } C_2 = 2$$

$$s(t) = \tfrac{at^2}{2} + 2$$

$$0 = s(1.57) = \frac{a \cdot (1.57)^2}{2} + 2$$

$$a = \frac{2 \cdot (-2)}{(1.57)^2}$$

$$\boxed{a \approx -1.62\mathrm{m}/\mathrm{s}^2}$$