Math 152.02
Calculus with Analytic Geometry II

January 26, 2011
1 Lecture 9 - 1/24
The Fundamental Theorem of Calculus II

2 Lecture 10 - 1/26
Techniques of Integration: Substitution
Proposition 69 (Motion with constant acceleration)

If an object has initial position \( s_0 \), initial velocity \( v_0 \), and constant acceleration \( g \) then the position of the object at time \( t \) is

\[
s(t) = \frac{gt^2}{2} + v_0 t + s_0
\]

Proof.

We have the following diff. eq. and initial conditions

\[
s'' = g, \quad s'(0) = v_0, \quad s(0) = s_0
\]

\[
s' = \int g \, dt
\]

\[
= gt + C_1
\]

\[
v_0 = s'(0) = g \cdot 0 + C_1
\]

\[
s = \int (gt + v_0) \, dt
\]

\[
= \frac{gt^2}{2} + v_0 t + C_2
\]

\[
s_0 = \frac{g \cdot 0^2}{2} + v_0 \cdot 0 + C_2
\]

so \( C_2 = s_0 \)

\[
\]

so \( C_1 = v_0 \)

\[
s(t) = \frac{gt^2}{2} + v_0 t + s_0
\]
Problem 70

*Find the initial velocity of an object thrown upward from the ground if it reaches a height of 100m in 10s. Assume that the acceleration of gravity is \(-9.8\text{m/s}^2\).*

Solution to Problem 70

Height at time \(t\) is

\[
s(t) = -\frac{9.8 \cdot t^2}{2} + v_0 t + 0
\]

Height at time 10 is

\[
100 = s(10) = -\frac{9.8 \cdot 10^2}{2} + v_0 \cdot 10
\]

So velocity at time \(t = 0\) is

\[
v_0 = \frac{100 + \frac{9.8 \cdot 10^2}{2}}{10} = \boxed{59\text{m/s}}
\]
The Fundamental Theorem of Calculus II

Recall that a function is a rule takes an element $x$ of the domain and returns a real number $f(x)$.

Some functions can be specified with a formula e.g.

- $f(x) = \sqrt{x^2 + 15e^x}$
- $g(x) = x \sin(x)$

Many function exist but have no nice formula description e.g.

- $p(t) =$ price of a share of Google stock at time $t$
- $r(t) =$ number of cars on the road in the US at time $t$
Some functions are naturally integrals of other functions e.g.

- If $c(x)$ is number of cars sold per day $x$ days after midnight Dec. 31, 2010
- then the number of cars sold this year $x$ days into 2011 is

$$T(x) = \int_0^x c(t) \, dt$$

- If $c$ has a formula, say $c(x) = \sqrt{20(10000 + \ln x)}$
- then we have a formula for $T$

$$T(x) = \int_0^x \sqrt{20(10000 + \ln t)} \, dt$$

- This is a formula for $T$!
- It may or may not simplify nicely using our antiderivative rules, but it is a formula.
Definition 71 (Elementary function)

A function is **elementary** if it has a formula coming from addition, subtraction, products, quotients, or compositions of polynomials, trigonometric functions, inverse trigonometric functions, $e^x$ and, $\ln x$

Example 72

\[
q(x) = \frac{e^{\sqrt{x^7 + \pi}} \cdot \arcsin(x^3)}{12 \ln(x^3) - 10 + e^x \ln x}
\]

is an elementary function
Example 73

The following formula for \( F(x) \) simplifies as follows

\[
F(x) = \int_3^x \cos t + t^3 \, dt
\]

\[
= \sin t + \frac{t^4}{4} \bigg|_3^x
\]

\[
= \left( \sin x + \frac{x^4}{4} \right) - \left( \sin 3 + \frac{3^4}{4} \right)
\]

\[
= \sin x + \frac{x^4}{4} - \sin 3 - \frac{81}{4}
\]

So \( F \) is an elementary function.
Example 74

1. \[ G(x) = \int_{0}^{x} \sec^3 t \, dt \]
   is difficult to simplify, but is in fact an elementary function

2. \[ H(x) = \int_{0}^{x} e^{-t^2} \, dt \]
   is impossible to simplify and is not an elementary function
Example 75

If $G(x) = \int_{2}^{x} t^2 + 7e^t \, dt$ then

$$\frac{d}{dx} G(x) = \frac{d}{dx} \left( \int_{2}^{x} t^2 + 7e^t \, dt \right)$$

$$= \frac{d}{dx} \left( \frac{t^3}{3} + 7e^t \bigg|_{2}^{x} \right)$$

$$= \frac{d}{dx} \left( \frac{x^3}{3} + 7e^x - \frac{2^3}{3} - 7e^2 \right)$$

$$= x^2 + 7e^x$$

Notice that this is the function we integrated in the first place.

Theorem 76 (Fundamental Theorem of Calculus II)

If $f$ is continuous and $a$ is a constant then

$$f(x) = \frac{d}{dx} \int_{a}^{x} f(t) \, dt$$
Problem 77

Simplify the following

1. \( \frac{d}{dx} \int_3^x 8 \arctan(t^3) \, dt \)
2. \( \frac{d}{dx} \int_{\pi}^x e^t \cot(t^2) \, dt \)

Solution to Problem 77

1. \( \frac{d}{dx} \int_3^x 8 \arctan(t^3) \, dt = 8 \arctan(x^3) \)

2. \( \frac{d}{dx} \int_{\pi}^x e^t \cot(t^2) \, dt = e^x \cot(x^2) \)
Note: FTOC II guarantees that every continuous function has an antiderivative

### Example 78

1. Find a formula for an antiderivative of $f(x) = \cos(x^2)$.

   Solution: $F(x) = \int_3^x \cos(t^2) \, dt$

2. Find a formula for an antiderivative of $g(u) = \ln(u^2 + 1)$.

   Solution: $G(u) = \int_{20}^{u} \ln(x^2 + 1) \, dx$

3. Find a formula for an antiderivative of $h(t) = \frac{e^t}{1+t^2}$.

   Solution: $H(t) = \int_0^t \frac{e^x}{1 + x^2} \, dx$
Example 79 (Combining FTOC II and chain rule)

Suppose

\[ F(x) = \int_{6}^{x^4 + e^x} \cos(t^2) \, dt \]

and we want a simplified expression for

\[ \frac{d}{dx} F(x). \]

If we let \( G(x) = \int_{6}^{x} \cos(t^2) \, dt \) and \( H(x) = x^4 + e^x \) then

\[ F(x) = G(H(x)) \]

By FTOC II

\[ G'(x) = \cos(x^2) \]

so by the chain rule

\[ F'(x) = G'(H(x)) \cdot H'(x) \]

\[ = \cos \left( [H(x)]^2 \right) \cdot H'(x) \]

\[ = \cos \left( (x^4 + e^x)^2 \right) \cdot (4x^3 + e^x) \]
Problem 80

Simplify the following

1. \[ \frac{d}{dx} \int_{6}^{x \sin x} 3t^2 \ln(\cos(t^3)) \, dt \]
2. \[ \frac{d}{dx} \int_{0}^{\arctan x} t^2 \sqrt{t + 200} \, dt \]

Solution to Problem 80

Using same technique as Example 79

1. \[
\begin{align*}
\frac{d}{dx} \int_{6}^{x \sin x} 3t^2 \ln(\cos(t^3)) \, dt &= 3(x \sin x)^2 \ln(\cos((x \sin x)^3)) \cdot (x \cos x + \sin x) \\
\end{align*}
\]

2. \[
\begin{align*}
\frac{d}{dx} \int_{0}^{\arctan x} t^2 \sqrt{t + 200} \, dt &= \left( (\arctan x)^2 \sqrt{\arctan x + 200} \right) \cdot \frac{1}{1+x^2} \\
\end{align*}
\]
Example 81 (Manipulation before applying FTOC II)

Suppose

$$F(x) = \int_{x^5}^{x^3 \cos x} t^4 \sin t \, dt$$

and we want a simplified expression for

$$\frac{d}{dx} F(x).$$

We have

$$F(x) = \int_{x^5}^{x^3 \cos x} t^4 \sin t \, dt = \int_{0}^{x^5} t^4 \sin t \, dt + \int_{0}^{x^3 \cos x} t^4 \sin t \, dt \quad (Theorem 30)$$

$$= -\int_{0}^{x^5} t^4 \sin t \, dt + \int_{0}^{x^3 \cos x} t^4 \sin t \, dt$$

Now we can apply FTOC II and chain rule separately to each integral

$$\frac{d}{dx} F(x) = -(x^5)^4 \sin(x^5) \cdot (5x^4)$$

$$+ (x^3 \cos x)^4 \sin(x^3 \cos x) \cdot (3x^2 \cos x - x^3 \sin x)$$
Problem 82

Simplify the following

\[
\frac{d}{du} \int_{\ln u}^{u^6} \frac{x}{x^3 + 1} \, dx
\]

Solution to Problem 82

Using same technique as Example 81

\[
\frac{d}{du} \int_{\ln u}^{u^6} \frac{x}{x^3 + 1} \, dx
\]

\[
= \frac{d}{du} \left( \int_{\ln u}^{0} \frac{x}{x^3 + 1} \, dx + \int_{0}^{u^6} \frac{x}{x^3 + 1} \, dx \right) \quad (\text{Theorem 30})
\]

\[
= \frac{d}{du} \left( -\int_{0}^{\ln u} \frac{x}{x^3 + 1} \, dx + \int_{0}^{u^6} \frac{x}{x^3 + 1} \, dx \right)
\]

\[
= -\frac{\ln u}{(\ln u)^3 + 1} \cdot \frac{1}{u} + \frac{u^6}{(u^6)^3 + 1} \cdot 6u^5
\]
Recall that the chain rule says that if

$$f(x) = \frac{d}{dx} F(x)$$

then for any differentiable function $g$

$$f(g(x)) \cdot g'(x) = \frac{d}{dx} F(g(x))$$

Interpreted as an antidifferentiation rule we get

**Theorem 83 (Integration by Substitution)**

*Suppose*

$$\int f(u) \, du = F(u) + C$$

*Then for any differentiable function $g$*

$$\int f(g(x)) \cdot g'(x) \, dx = F(g(x)) + C$$
Steps for substitution in practice

Given an integral e.g.
\[ \int x \cos (x^2) \, dx \]

1. Choose \( u \) to be a function whose derivative is (up to multiplication by a constant) a factor of your integrand
   
   e.g. \( u = x^2 \)

2. Compute \( \frac{du}{dx} \) and solve for the differential \( du \)
   
   e.g. \( \frac{du}{dx} = 2x \)
   
   so \( du = 2x \, dx \)

3. Rearrange the integral until it is of the form \( \int f(u) \, du \) for some function \( f \).
   
   e.g. \( \int x \cos (x^2) \, dx = \int \cos (x^2) \cdot \frac{1}{2} \cdot 2x \, dx = \int \frac{1}{2} \cos u \, du \)

   If this is impossible return to step 1 with a new choice for \( u \)
Steps for substitution in practice (*continued*)

4 Evaluate the antiderivative $\int f(u) \, du$

\[ \begin{align*} 
\text{e.g.} & \quad \int \frac{1}{2} \cos u \, du = \frac{1}{2} \sin u + C \\
\end{align*} \]

5 Replace $u$ with function of $x$ from step 1

\[ \begin{align*} 
\text{e.g.} & \quad \frac{1}{2} \sin u + C = \frac{1}{2} \sin(x^2) + C \\
\end{align*} \]

6 Check your work

\[ \begin{align*} 
\text{e.g.} \quad \frac{d}{dx} \left( \frac{1}{2} \sin(x^2) \right) = \frac{1}{2} \cos(x^2) \cdot 2x = x \cos(x^2) \quad \checkmark \\
\end{align*} \]
Problem 84

Evaluate $\int \tan x \, dx$

Solution to Problem 84

\[
\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx
\]

\[u = \cos x, \quad du = -\sin x \, dx\]

\[= \int -\frac{1}{\cos x} \cdot (-\sin x) \, dx\]

\[= -\int \frac{1}{u} \, du\]

\[= -\ln |u| + C\]

\[= -\ln |\cos x| + C\]

Check your work

\[
\frac{d}{dx} \left( -\ln |\cos x| \right) = -\frac{1}{\cos x} \cdot (-\sin x) = \frac{\sin x}{\cos x} = \tan x \checkmark
\]
Problem 85

Evaluate \( \int \frac{t^5}{1+t^6} \, dt \)

Solution to Problem 85

\[
\int \frac{t^5}{1+t^6} \, dt = 
\]

\[
= \int \frac{1}{1+t^6} \cdot \frac{1}{6} \cdot 6t^5 \, dt 
\]

\[
= \frac{1}{6} \int \frac{1}{u} \, du 
\]

\[
= \frac{1}{6} \ln |u| + C 
\]

\[
= \frac{1}{6} \ln |1 + t^6| + C 
\]

Check your work

\[
\frac{d}{dt} \left( \frac{1}{6} \ln |1 + t^6| \right) = \frac{1}{6} \cdot \frac{1}{1+t^6} \cdot (6t^5) = \frac{t^5}{1+t^6} \quad \checkmark 
\]
Problem 86

Evaluate $\int \frac{e^y}{1+e^{2y}} \, dy$

Solution to Problem 86

\[
\int \frac{e^y}{1+e^{2y}} \, dy = 
\]
\[
u = e^y, \quad du = e^y \, dy \\
= \int \frac{1}{1+(e^y)^2} \cdot e^y \, dy \\
= \int \frac{1}{1+u^2} \, du \\
= \arctan u + C \\
= \arctan(e^y) + C
\]

Check your work

\[
\frac{d}{dy} \left( \arctan(e^y) \right) = \frac{1}{1+(e^y)^2} \cdot e^y = \frac{e^y}{1+e^{2y}} \quad \checkmark
\]
Linear substitutions

- Sometimes a linear substitution of $u = mx + b$ simplifies an integral.
- This substitution will always work since $du = mdx$ and you can replace all $x$’s with $\frac{u-b}{m}$

Example 87

We may substitute $u = 4x$ and hence $du = 4dx$ into

$$\int e^{4x} \, dx = \int e^{4x} \cdot \frac{1}{4} \cdot 4 \, dx = \frac{1}{4} \int e^u \, du = \frac{1}{4} e^u + C = \frac{1}{4} e^{4x} + C$$
Problem 88

Evaluate $\int x^2 \sqrt{x - 2} \, dx$

Solution to Problem 88

$$\int x^2 \sqrt{x - 2} \, dx =$$

$$u = x - 2, \quad du = dx, \quad x = u + 2$$

$$= \int (u + 2)^2 \sqrt{u} \, du$$

$$= \int (u^2 + 4u + 4)u^{\frac{1}{2}} \, du$$

$$= \int u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}} \, du$$

$$= \frac{2}{7}u^{\frac{7}{2}} + \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}} + C$$

$$= \frac{2}{7}(x - 2)^{\frac{7}{2}} + \frac{8}{5}(x - 2)^{\frac{5}{2}} + \frac{8}{3}(x - 2)^{\frac{3}{2}} + C$$

Check your work
Example 89 (Substitution in definite integrals)

Evaluate

\[
\int_{2}^{5} \frac{t^2}{7 + t^3} \, dt
\]

\[
\int_{2}^{5} \frac{t^2}{7 + t^3} \, dt =
\]

\[
u = 7 + t^3, \quad du = 3t^2 \, dt,
\]

\[
= \int_{7+2^3}^{7+5^3} \frac{1}{3} \cdot \frac{1}{u} \, du \quad \text{(note: limits changed)}
\]

\[
= \frac{1}{3} \ln |u| \bigg|_{132}^{15}
\]

\[
= \frac{1}{3} \ln(132) - \frac{1}{3} \ln(15)
\]

Notice we did not substitute \(7 + t^3\) back in for \(u\).
Example 90 (Substitution in definite integrals (alt. sol.))

Evaluate

\[ \int_2^5 \frac{t^2}{7 + t^3} \, dt \]

\[ \int \frac{t^2}{7 + t^3} \, dt = \]

\[ u = 7 + t^3, \quad du = 3t^2 \, dt, \]

\[ = \int \frac{1}{3} \cdot \frac{1}{u} \cdot 3t^2 \, dt \]

\[ = \int \frac{1}{3} \cdot \frac{1}{u} \, du \]

\[ = \frac{1}{3} \ln |u| + C \]

\[ = \frac{1}{3} \ln |7 + t^3| + C \]

\[ \int_2^5 \frac{t^2}{7 + t^3} \, dt = \left. \frac{1}{3} \ln |7 + t^3| \right|_2^5 = \frac{1}{3} \ln(132) - \frac{1}{3} \ln(15) \]
Problem 91

Evaluate

\[ \int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin \sqrt{u}}{\sqrt{u}} \, du \]

Solution to Problem 91

\[
\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin \sqrt{u}}{\sqrt{u}} \, du = \\
\quad w = \sqrt{u}, \quad dw = \frac{1}{2\sqrt{u}} \, du, \\
= \int_{\frac{\pi^2}{4}}^{\frac{\pi^2}{4}} 2 \sin \sqrt{u} \cdot \frac{1}{2\sqrt{u}} \, du \\
= \int_{\sqrt{\frac{\pi^2}{4}}}^{\sqrt{\pi^2}} 2 \sin w \, dw \\
= -2 \cos w \bigg|_{\frac{\pi}{2}}^{\pi} = (-2 \cos \pi) - (-2 \cos \frac{\pi}{2}) = 2
\]