Math 152.02
Calculus with Analytic Geometry II

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Proposition 69 (Motion with constant acceleration)

If an object has **initial position** s_0, **initial velocity** v_0, *and* **constant acceleration** g *then the position of the object at time* t *is*

$$s(t) = \frac{gt^2}{2} + v_0 t + s_0$$

Proof.

We have the following diff. eq. and initial conditions

$$s'' = g, s'(0) = v_0, s(0) = s_0$$

$$s' = \int g \, \mathrm{d}t$$
$$= gt + C_1$$

$$v_0 = s'(0) = g \cdot 0 + C_1$$

so $C_1 = v_0$

$$s = \int gt + v_0 \, \mathrm{d}t$$
$$= \frac{gt^2}{2} + v_0 t + C_2$$

$$s_0 = \frac{g \cdot 0^2}{2} + v_0 \cdot 0 + C_2$$

so $C_2 = s_0$

$$s(t) = \frac{gt^2}{2} + v_0 t + s_0$$

□

Problem 70

Find the intial velocity of an object thrown upward from the ground if it reaches a height of $100\mathrm{m}$ *in* $10\mathrm{s}$. *Assume that the acceleration of gravity is* $-9.8\mathrm{m/s}^2$.

Solution to Problem 70

Height at time t is

$$s(t) = -\frac{9.8 \cdot t^2}{2} + v_0 t + 0$$

Height at time 10 is

$$100 = s(10) = -\frac{9.8 \cdot 10^2}{2} + v_0 \cdot 10$$

So velocity at time $t = 0$ is

$$v_0 = \frac{100 + \frac{9.8 \cdot 10^2}{2}}{10} = \boxed{59\mathrm{m/s}}$$

The Fundamental Theorem of Calculus II

Recall that a **function** is a rule takes an element x of the **domain** and returns a real number $f(x)$.

Some functions can be specified with a **formula** *e.g.*

- $f(x) = \sqrt{x^2 + 15e^x}$
- $g(x) = x\sin(x)$

Many function exist but have no nice formula description *e.g.*

- $p(t) =$ price of a share of Google stock at time t
- $r(t) =$ number of cars on the road in the US at time t

Some functions are naturally integrals of other functions *e.g.*

- If $c(x)$ is number of cars sold per day x days after midnight Dec. 31, 2010
- then the number of cars sold this year x days into 2011 is

$$T(x) = \int_0^x c(t)\,\mathrm{d}t$$

- If c has a formula, say $c(x) = \sqrt{20(10000 + \ln x)}$
- then we have a formula for T

$$T(x) = \int_0^x \sqrt{20(10000 + \ln t)}\,\mathrm{d}t$$

- This is a formula for T!
- It may or may not simplify nicely using our antiderivative rules, but it is a formula.

Definition 71 (Elementary function)

A function is **elementary** if it has a formula coming from addition, subtraction, products, quotients, or compositions of polynomials, trigonometric functions, inverse trignonometric functions, e^x and, $\ln x$

Example 72

$$q(x) = \frac{e^{\sqrt{x^7+\pi}} \cdot \arcsin(x^3)}{\sqrt[12]{\ln(x^3) - 10 + e^{x \ln x}}}$$

is an elementary function

Example 73

The following formula for $F(x)$ simplifies as follows

$$\begin{aligned} F(x) &= \int_3^x \cos t + t^3 \, \mathrm{d}t \\ &= \sin t + \frac{t^4}{4}\Big|_3^x \\ &= \left(\sin x + \frac{x^4}{4}\right) - \left(\sin 3 + \frac{3^4}{4}\right) \\ &= \sin x + \frac{x^4}{4} - \sin 3 - \frac{81}{4} \end{aligned}$$

So F is an elementary function

Example 74

1. $$G(x) = \int_0^x \sec^3 t \, \mathrm{d}t$$

   is difficult to simplify, but is in fact an elementary funtion

2. $$H(x) = \int_0^x e^{-t^2} \, \mathrm{d}t$$

   is impossible to simplify and is not an elementary function

Example 75

If $G(x) = \int_2^x t^2 + 7e^t \, \mathrm{d}t$ then

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x} G(x) &= \tfrac{\mathrm{d}}{\mathrm{d}x} \left(\int_2^x t^2 + 7e^t \, \mathrm{d}t \right) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}x} \left(\frac{t^3}{3} + 7e^t \Big|_2^x \right) \\
&= \tfrac{\mathrm{d}}{\mathrm{d}x} \left(\frac{x^3}{3} + 7e^x - \frac{2^3}{3} - 7e^2 \right) \\
&= x^2 + 7e^x
\end{aligned}$$

Notice that this is the function we integrated in the first place.

Theorem 76 (Fundamental Theorem of Calculus II)

If f is continuous and a is a constant then

$$f(x) = \tfrac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t) \, \mathrm{d}t$$

Problem 77

Simplify the following

1. $\frac{\mathrm{d}}{\mathrm{d}x}\int_3^x 8\arctan(t^3)\,\mathrm{d}t$
2. $\frac{\mathrm{d}}{\mathrm{d}x}\int_\pi^x e^t\cot(t^2)\,\mathrm{d}t$

Solution to Problem 77

1. $$\frac{\mathrm{d}}{\mathrm{d}x}\int_3^x 8\arctan(t^3)\,\mathrm{d}t = \boxed{8\arctan(x^3)}$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x}\int_\pi^x e^t\cot(t^2)\,\mathrm{d}t = \boxed{e^x\cot(x^2)}$$

Note: FTOC II guarantees that **every continuous function has an antiderivative**

Example 78

1. Find a formula for an antiderivative of $f(x) = \cos(x^2)$.

$$\text{Solution:} \quad F(x) = \int_3^x \cos(t^2)\,\mathrm{d}t$$

2. Find a formula for an antiderivative of $g(u) = \ln(u^2 + 1)$.

$$\text{Solution:} \quad G(u) = \int_{20}^u \ln(x^2 + 1)\,\mathrm{d}x$$

3. Find a formula for an antiderivative of $h(t) = \frac{e^t}{1+t^2}$.

$$\text{Solution:} \quad H(t) = \int_0^t \frac{e^x}{1 + x^2}\,\mathrm{d}x$$

Example 79 (Combining FTOC II and chain rule)

Suppose

$$F(x) = \int_6^{x^4+e^x} \cos(t^2)\,\mathrm{d}t$$

and we want a simplified expression for

$$\tfrac{\mathrm{d}}{\mathrm{d}x} F(x).$$

If we let $G(x) = \int_6^x \cos(t^2)\,\mathrm{d}t$ and $H(x) = x^4 + e^x$ then

$$F(x) = G(H(x))$$

By FTOC II

$$G'(x) = \cos(x^2)$$

so by the chain rule

$$\begin{aligned} F'(x) &= G'(H(x)) \cdot H'(x) \\ &= \cos\left([H(x)]^2\right) \cdot H'(x) \\ &= \cos\left((x^4 + e^x)^2\right) \cdot (4x^3 + e^x) \end{aligned}$$

Problem 80

Simplify the following

1. $\frac{\mathrm{d}}{\mathrm{d}x}\int_6^{x\sin x} 3t^2\ln(\cos(t^3))\,\mathrm{d}t$
2. $\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\arctan x} t^2\sqrt{t+200}\,\mathrm{d}t$

Solution to Problem 80

Using same technique as Example 79

1. $$\frac{\mathrm{d}}{\mathrm{d}x}\int_6^{x\sin x} 3t^2\ln(\cos(t^3))\,\mathrm{d}t$$
$$= \boxed{3(x\sin x)^2\ln(\cos((x\sin x)^3))\cdot(x\cos x+\sin x)}$$

2. $$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\arctan x} t^2\sqrt{t+200}\,\mathrm{d}t$$
$$= \boxed{\left((\arctan x)^2\sqrt{\arctan x+200}\right)\cdot\frac{1}{1+x^2}}$$

Example 81 (Manipulation before applying FTOC II)

Suppose

$$F(x) = \int_{x^5}^{x^3 \cos x} t^4 \sin t \, \mathrm{d}t$$

and we want a simplified expression for

$$\tfrac{\mathrm{d}}{\mathrm{d}x} F(x).$$

$$\begin{aligned} F(x) &= \int_{x^5}^{x^3 \cos x} t^4 \sin t \, \mathrm{d}t \\ &= \int_{x^5}^{0} t^4 \sin t \, \mathrm{d}t + \int_{0}^{x^3 \cos x} t^4 \sin t \, \mathrm{d}t \quad \textit{(Theorem 30)} \\ &= -\int_{0}^{x^5} t^4 \sin t \, \mathrm{d}t + \int_{0}^{x^3 \cos x} t^4 \sin t \, \mathrm{d}t \end{aligned}$$

Now we can apply FTOC II and chain rule separately to each integral

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x} F(x) = &-(x^5)^4 \sin(x^5) \cdot (5x^4) \\ &+ (x^3 \cos x)^4 \sin(x^3 \cos x) \cdot (3x^2 \cos x - x^3 \sin x) \end{aligned}$$

Problem 82

Simplify the following

$$\frac{\mathrm{d}}{\mathrm{d}u}\int_{\ln u}^{u^6}\frac{x}{x^3+1}\,\mathrm{d}x$$

Solution to Problem 82

Using same technique as Example 81

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}u}\int_{\ln u}^{u^6}\frac{x}{x^3+1}\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(\int_{\ln u}^{0}\frac{x}{x^3+1}\,\mathrm{d}x + \int_{0}^{u^6}\frac{x}{x^3+1}\,\mathrm{d}x\right) \qquad \textit{(Theorem 30)} \\
&= \frac{\mathrm{d}}{\mathrm{d}u}\left(-\int_{0}^{\ln u}\frac{x}{x^3+1}\,\mathrm{d}x + \int_{0}^{u^6}\frac{x}{x^3+1}\,\mathrm{d}x\right) \\
&= \boxed{-\frac{\ln u}{(\ln u)^3+1}\cdot\frac{1}{u} + \frac{u^6}{(u^6)^3+1}\cdot 6u^5}
\end{aligned}$$

Techniques of Integration: Substitution

Recall that the chain rule says that if

$$f(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(x)$$

then for any differentiable function g

$$f(g(x)) \cdot g'(x) = \frac{\mathrm{d}}{\mathrm{d}x} F(g(x))$$

Interpreted as an antidifferentiation rule we get

Theorem 83 (Integration by Substitution)

Suppose

$$\int f(u)\,\mathrm{d}u = F(u) + C$$

Then for any differentiable function g

$$\int f(g(x)) \cdot g'(x)\,\mathrm{d}x = F(g(x)) + C$$

Steps for substitution in practice

Given an integral *e.g.*

$$\int x\cos\left(x^2\right)\mathrm{d}x$$

1. Choose u to be a function whose derivative is (up to multiplication by a constant) a factor of your integrand

$$e.g. \quad u = x^2$$

2. Compute $\frac{\mathrm{d}u}{\mathrm{d}x}$ and solve for the differential $\mathrm{d}u$

$$\begin{aligned} e.g. \quad \frac{\mathrm{d}u}{\mathrm{d}x} &= 2x \\ \text{so} \quad \mathrm{d}u &= 2x\,\mathrm{d}x \end{aligned}$$

3. Rearrange the integral until it is of the form $\int f(u)\,\mathrm{d}u$ for some function f.

$$e.g. \int x\cos\left(x^2\right)\mathrm{d}x = \int \cos\overbrace{\left(x^2\right)}^{u} \cdot \frac{1}{2} \cdot \overbrace{2x\,\mathrm{d}x}^{\mathrm{d}u} = \int \frac{1}{2}\cos u\,\mathrm{d}u$$

If this is impossible return to step 1 with a new choice for u

Steps for substitution in practice *(continued)*

4. Evaluate the antiderivative $\int f(u)\,\mathrm{d}u$

$$e.g. \quad \int \frac{1}{2}\cos u\,\mathrm{d}u = \frac{1}{2}\sin u + C$$

5. Replace u with function of x from step 1

$$e.g. \quad \frac{1}{2}\sin u + C = \boxed{\frac{1}{2}\sin(x^2) + C}$$

6. Check your work

$$e.g. \quad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\sin(x^2)\right) = \frac{1}{2}\cos(x^2)\cdot 2x = x\cos(x^2) \checkmark$$

Problem 84

Evaluate $\int \tan x \, \mathrm{d}x$

Solution to Problem 84

$$\begin{aligned}
\int \tan x \, \mathrm{d}x &= \int \frac{\sin x}{\cos x} \, \mathrm{d}x \\
&\qquad u = \cos x, \quad \mathrm{d}u = -\sin x \, \mathrm{d}x \\
&= \int -\frac{1}{\cos x} \cdot (-\sin x) \, \mathrm{d}x \\
&= -\int \frac{1}{u} \, \mathrm{d}u \\
&= -\ln|u| + C \\
&= \boxed{-\ln|\cos x| + C}
\end{aligned}$$

Check your work

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(-\ln|\cos x|\Big) = \frac{-1}{\cos x} \cdot (-\sin x) = \frac{\sin x}{\cos x} = \tan x \checkmark$$

Problem 85

Evaluate $\int \frac{t^5}{1+t^6}\,\mathrm{d}t$

Solution to Problem 85

$$\int \frac{t^5}{1+t^6}\,\mathrm{d}t =$$

$$u = 1 + t^6, \quad \mathrm{d}u = 6t^5\,\mathrm{d}t$$

$$= \int \frac{1}{1+t^6} \cdot \tfrac{1}{6} \cdot 6t^5\,\mathrm{d}t$$

$$= \tfrac{1}{6} \int \frac{1}{u}\,\mathrm{d}u$$

$$= \tfrac{1}{6} \ln|u| + C$$

$$= \boxed{\tfrac{1}{6} \ln|1+t^6| + C}$$

Check your work

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{6}\ln|1+t^6|\right) = \frac{1}{6} \cdot \frac{1}{1+t^6} \cdot (6t^5) = \frac{t^5}{1+t^6} \checkmark$$

Problem 86

Evaluate $\int \frac{e^y}{1+e^{2y}}\,\mathrm{d}y$

Solution to Problem 86

$$\int \frac{e^y}{1+e^{2y}}\,\mathrm{d}y =$$

$$u = e^y, \quad \mathrm{d}u = e^y\,\mathrm{d}y$$

$$= \int \frac{1}{1+(e^y)^2}\cdot e^y\,\mathrm{d}y$$

$$= \int \frac{1}{1+u^2}\,\mathrm{d}u$$

$$= \arctan u + C$$

$$= \boxed{\arctan(e^y) + C}$$

Check your work

$$\frac{\mathrm{d}}{\mathrm{d}y}\Big(\arctan(e^y)\Big) = \frac{1}{1+(e^y)^2}\cdot e^y = \frac{e^y}{1+e^{2y}} \checkmark$$

Linear substitutions

- Sometimes a linear substitution of $u = mx + b$ simplfies an integral.
- This substitution will always work since $\mathrm{d}u = m\,\mathrm{d}x$ and you can replace all x's with $\frac{u-b}{m}$

Example 87

We may substitute $u = 4x$ and hence $\mathrm{d}u = 4\,\mathrm{d}x$ into

$$\int e^{4x}\,\mathrm{d}x = \int e^{4x} \cdot \tfrac{1}{4} \cdot 4\,\mathrm{d}x = \tfrac{1}{4}\int e^{u}\,\mathrm{d}u = \tfrac{1}{4}e^{u} + C = \tfrac{1}{4}e^{4x} + C$$

Problem 88

Evaluate $\int x^2\sqrt{x-2}\,\mathrm{d}x$

Solution to Problem 88

$$\int x^2\sqrt{x-2}\,\mathrm{d}x =$$

$$u = x-2, \quad \mathrm{d}u = \mathrm{d}x, \quad x = u+2$$

$$= \int (u+2)^2\sqrt{u}\,\mathrm{d}u$$

$$= \int (u^2+4u+4)u^{\frac{1}{2}}\,\mathrm{d}u$$

$$= \int u^{\frac{5}{2}} + 4u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\,\mathrm{d}u$$

$$= \frac{2}{7}u^{\frac{7}{2}} + \frac{2}{5}\cdot 4u^{\frac{5}{2}} + \frac{2}{3}\cdot 4u^{\frac{3}{2}} + C$$

$$= \boxed{\frac{2}{7}(x-2)^{\frac{7}{2}} + \frac{8}{5}(x-2)^{\frac{5}{2}} + \frac{8}{3}(x-2)^{\frac{3}{2}} + C}$$

Check your work

Example 89 (Substitution in definite integrals)

Evaluate

$$\int_2^5 \frac{t^2}{7+t^3}\,\mathrm{d}t$$

$$\int_2^5 \frac{t^2}{7+t^3}\,\mathrm{d}t =$$

$$u = 7 + t^3, \quad \mathrm{d}u = 3t^2\,\mathrm{d}t,$$

$$= \int_2^5 \frac{1}{3}\cdot\frac{1}{7+t^3}\cdot 3t^2\,\mathrm{d}t$$

$$= \int_{7+2^3}^{7+5^3} \frac{1}{3}\cdot\frac{1}{u}\,\mathrm{d}u \qquad \text{(note: limits changed)}$$

$$= \frac{1}{3}\ln|u|\Big|_{15}^{132}$$

$$= \boxed{\tfrac{1}{3}\ln(132) - \tfrac{1}{3}\ln(15)}$$

Notice we did **not** substitute $7 + t^3$ back in for u.

Example 90 (Substitution in definite integrals (alt. sol.))

Evaluate

$$\int_2^5 \frac{t^2}{7+t^3}\,\mathrm{d}t$$

$$\int \frac{t^2}{7+t^3}\,\mathrm{d}t =$$

$$u = 7+t^3, \quad \mathrm{d}u = 3t^2\,\mathrm{d}t,$$

$$= \int \frac{1}{3}\cdot\frac{1}{7+t^3}\cdot 3t^2\,\mathrm{d}t$$

$$= \int \frac{1}{3}\cdot\frac{1}{u}\,\mathrm{d}u$$

$$= \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|7+t^3| + C$$

$$\int_2^5 \frac{t^2}{7+t^3}\,\mathrm{d}t = \frac{1}{3}\ln|7+t^3|\Big|_2^5 = \boxed{\tfrac{1}{3}\ln(132) - \tfrac{1}{3}\ln(15)}$$

Problem 91

Evaluate

$$\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin\sqrt{u}}{\sqrt{u}}\,\mathrm{d}u$$

Solution to Problem 91

$$\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\sin\sqrt{u}}{\sqrt{u}}\,\mathrm{d}u =$$

$$w = \sqrt{u}, \quad \mathrm{d}w = \frac{1}{2\sqrt{u}}\,\mathrm{d}u,$$

$$= \int_{\frac{\pi^2}{4}}^{\pi^2} 2\sin\sqrt{u}\cdot\frac{1}{2\sqrt{u}}\,\mathrm{d}u$$

$$= \int_{\sqrt{\frac{\pi^2}{4}}}^{\sqrt{\pi^2}} 2\sin w\,\mathrm{d}w$$

$$= -2\cos w\Big|_{\frac{\pi}{2}}^{\pi} = (-2\cos\pi) - (-2\cos\frac{\pi}{2}) = \boxed{2}$$