Math 152.02
Calculus with Analytic Geometry II

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More basic integrals

- $$\int \tan x \, dx = -\ln|\cos x| + C$$
- $$\int \cot x \, dx = \ln|\sin x| + C$$
- $$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$
- $$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$
- $$\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln|1 + x^2| + C$$

Trig integrals

Trig integrals

For integrals of the following form

$$\int \sin^m x \cos^n x \, dx$$

CASE I: $n > 0$ odd
Substitute $u = \sin x$

CASE II: $m > 0$ odd
Substitute $u = \cos x$

CASE III: $n > 0$ and $m > 0$ both even
Use half angle formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$
$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Example 92

Compute $\int \sin^5 x \cos^{12} x \, dx$

$$\int \sin^5 x \cos^{12} x \, dx = \qquad \textit{(apply CASE II)}$$

$$u = \cos x, \quad du = -\sin x \, dx, \quad \sin^2 x = 1 - u^2$$

$$= \int -\sin^4 x \cos^{12} x \cdot (-\sin x) \, dx$$

$$= \int -(\sin^2 x)^2 \cos^{12} x \cdot (-\sin x) \, dx$$

$$= \int -(1-u^2)^2 u^{12} \, du$$

$$= \int -u^{16} + 2u^{14} - u^{12} \, du$$

$$= -\frac{u^{17}}{17} + \frac{2u^{15}}{15} - \frac{u^{13}}{13} + C$$

$$= \boxed{-\frac{\cos^{17} x}{17} + \frac{2\cos^{15} x}{15} - \frac{\cos^{13} x}{13} + C}$$

Example 93

Compute $\int \sin^7 x \cos^3 x \, \mathrm{d}x$

$$\int \sin^7 x \cos^3 x \, \mathrm{d}x = \qquad \textit{(apply CASE I or II)}$$

$$u = \sin x, \quad \mathrm{d}u = \cos x \, \mathrm{d}x, \quad \cos^2 x = 1 - u^2$$

$$= \int \sin^7 x \cos^2 x \cdot (\cos x) \, \mathrm{d}x$$

$$= \int u^7 (1 - u^2) \, \mathrm{d}u$$

$$= \int u^7 - u^9 \, \mathrm{d}u$$

$$= \frac{u^8}{8} - \frac{u^{10}}{10} + C$$

$$= \boxed{-\frac{\sin^8 x}{8} - \frac{\sin^{10} x}{10} + C}$$

Example 94

Compute $\int \sin^7 8x \cos^{-2} 8x \, \mathrm{d}x$

$$\int \sin^7 8x \cos^{-2} 8x \, \mathrm{d}x = \qquad \textit{(apply CASE II)}$$

$$u = \cos 8x, \quad \mathrm{d}u = -8 \sin 8x \, \mathrm{d}x, \quad \sin^2 8x = 1 - u^2$$

$$= -\tfrac{1}{8} \int \sin^6 8x \cos^{-2} 8x \cdot (-8 \sin x) \, \mathrm{d}x$$

$$= -\tfrac{1}{8} \int (\sin^2 8x)^3 \cos^{-2} 8x \cdot (-8 \sin 8x) \, \mathrm{d}x$$

$$= -\tfrac{1}{8} \int (1 - u^2)^3 u^{-2} \, \mathrm{d}u$$

$$= -\tfrac{1}{8} \int u^{-2} - 3 + 3u^2 - u^4 \, \mathrm{d}u$$

$$= \tfrac{u^{-1}}{-8} + \tfrac{3}{8} u - \tfrac{3u^3}{3 \cdot 8} + \tfrac{u^5}{5 \cdot 8} + C$$

$$= \boxed{-\tfrac{\sec 8x}{8} + \tfrac{3}{8} \cos 8x - \tfrac{\cos^3 8x}{8} + \tfrac{\cos^5 8x}{40} + C}$$

Example 95

Compute $\int \sin^2 x \cos^4 x \, dx$

$$\int \sin^2 x \cos^4 x \, dx = \int \sin^2 x (\cos^2 x)^2 \, dx \quad \textit{(apply CASE III)}$$

$$= \int \frac{1 - \cos 2x}{2} \cdot \left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8} \int 1 + \cos 2x - \cos^2 2x - \cos^3 2x \, dx$$

$$= \frac{1}{8}x + \frac{1}{16} \sin 2x - \frac{1}{8} \int \cos^2 2x \, dx - \frac{1}{8} \int \cos^3 2x \, dx$$

$$= \frac{1}{8}x + \frac{1}{16} \sin 2x - \frac{1}{8} \int \frac{1 + \cos 4x}{2} \, dx - \frac{1}{8} \int \cos^3 2x \, dx$$

$$u = \sin 2x, \quad du = 2 \cos 2x \, dx$$

$$= \frac{1}{8}x + \frac{1}{16} \sin 2x - \frac{1}{16}x - \frac{1}{4 \cdot 16} \sin 4x - \frac{1}{16} \int 1 - u^2 \, du$$

$$= \frac{1}{8}x + \frac{1}{16} \sin 2x - \frac{1}{16}x - \frac{1}{64} \sin 4x - \frac{1}{16}u + \frac{u^3}{3 \cdot 16} + C$$

$$= \boxed{\frac{1}{8}x + \frac{1}{16} \sin 2x - \frac{1}{16}x - \frac{1}{64} \sin 4x - \frac{1}{16} \sin 2x + \frac{\sin^3 2x}{48} + C}$$

Trig integrals

For integrals of the following form

$$\int \tan^m x \sec^n x \, dx$$

CASE I: $n \geq 2$ even
Substitute $u = \tan x$

CASE II: $m > 0$ odd and $n > 0$
Substitute $u = \sec x$

CASE III: $n > 0$ odd and $m > 0$ even
Convert all $\tan^2 x$ to $(\sec^2 x - 1)$ and

$$\int \sec^m x \, dx = \frac{\sin x}{(m-1)\cos^{m-1} x} + \frac{m-2}{m-1} \int \sec^{m-2} x \, dx$$

Polynomials

Definition 96 (Polynomials and rational functions)

A **polynomial** is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

A **rational function** is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

Where p and q are polynomials.

Example 97

$$p(x) = 6x^8 - \pi x^5 + ex^3 + 1$$

is a polynomial.

Example 98

$$f(x) = \frac{2x^5 + 7x^4 + 2x^3 + 6x^2 + 2}{x^3 + 3x^2 + 5x + 1}$$

is a rational function.

Definition 99 (Degree of a polynomial)

If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and $a_n \neq 0$ then the **degree of** p is

$$\deg p = n.$$

Example 100

$$\deg(\pi x^5 - 6x^8 + e x^3 + 1) = 8$$

Polynomial Division

Example 101

Divide

$$\frac{x^4 + 3x + 2}{x^2 - 1}$$

$$\frac{x^4 + 3x + 2}{x^2 - 1} = x^2 + 1 + \frac{3x - 3}{x^2 - 1}$$

Example 102

Divide

$$\frac{2x^5 + 7x^4 + 2x^3 + 6x^2 + 2}{x^3 + 3x^2 + 5x + 1}$$

$$\frac{2x^5 + 7x^4 + 2x^3 + 6x^2 + 2}{x^3 + 3x^2 + 5x + 1} = 2x^2 + x - 11 + \frac{32x^2 + 54x + 9}{x^3 + 3x^2 + 5x + 1}$$

Completing the square

Completing the square

Given a quadratic polynomial

$$ax^2 + bx + c$$

it can be rewritten in the form

$$a(x + p)^2 + q$$

This process is called **completing the square**

Steps for completing the square

Given a degree 2 polynomial *e.g.*

$$5x^2 - 30x + 10$$

1. Factor out coefficient of x^2

$$\boxed{5}\left(x^2 - 6x + 2\right)$$

2. Divide coefficient of x by 2 square that number and add and subtract it.

$$5\left(x^2 - 6x + \boxed{9 - 9} + 2\right)$$

3. First terms are a perfect square

$$5\left(\boxed{(x-3)^2} - 9 + 2\right) = 5(x-3)^2 - 35$$

Example 103

Complete the square for $x^2 + 3x + 10$.

$$\begin{aligned} x^2 + 3x + 10 &= x^2 + 3x + \tfrac{9}{4} - \tfrac{9}{4} + 10 \\ &= \boxed{\left(x + \tfrac{3}{2}\right)^2 + \tfrac{31}{4}} \end{aligned}$$

Example 104

Complete the square for $2x^2 - 7x + 8$.

$$\begin{aligned} 2x^2 - 7x + 8 &= 2\left(x^2 - \tfrac{7}{2}x + 4\right) \\ &= 2\left(x^2 - \tfrac{7}{2}x + \tfrac{49}{16} - \tfrac{49}{16} + 4\right) \\ &= 2\left(\left(x - \tfrac{7}{4}\right)^2 + \tfrac{15}{16}\right) \\ &= \boxed{2\left(x - \tfrac{7}{4}\right)^2 + \tfrac{15}{8}} \end{aligned}$$

Factoring Polynomials

The quadratic equation

If $p(x) = ax^2 + bx + c$ then the roots of p are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$b^2 - 4ac$ is called the **discriminant** of p.

If the discriminant of p is negative then p does not factor. Otherwise, p factors as

$$p(x) = a\left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

Steps for integrating rational functions

Given an integral of a rational function *e.g.*

$$\int \frac{x^3+7x^2+6x-42}{x^2+2x-8}\,dx$$

1. If the degree of the numerator is $\geq$ the degree of the denominator

$$\text{e.g.} \quad \frac{x^3+7x^2+6x-42}{x^2+2x-8} = x+5+\frac{4x-2}{x^2+2x-8}$$

2. Factor the denominator

$$\text{e.g.} \quad x^2+2x-8=(x-2)(x+4)$$

3. Rewrite the fractional part as a sum of partial fractions

$$\text{e.g.} \quad \frac{4x-2}{x^2+2x-8}=\frac{A}{x-2}+\frac{B}{x+4}=\frac{1}{x-2}+\frac{3}{x+4}$$

Steps for integrating rational functions *(continued)*

4. Integrate

$$\int \frac{x^3+7x^2+6x-42}{x^2+2x-8}\,dx = \int x+5+\frac{4x-2}{x^2+2x-8}\,dx$$
$$= \int x+5+\frac{1}{x-2}+\frac{3}{x+4}\,dx$$
$$= \boxed{\frac{x^2}{2}+5x+\ln|x-2|+3\ln|x-4|+C}$$

Trig substitutions

Trig substitutions

For integrals involving $\sqrt{a^2 - x^2}$ it may help to substitute

$$x = a \sin \theta$$

For integrals involving $\sqrt{x^2 + a^2}$ it may help to substitute

$$x = a \tan \theta$$

For integrals involving $\sqrt{x^2 - a^2}$ it may help to substitute

$$x = a \sec \theta$$

Example 105

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

$$\begin{aligned}
\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x \\
& \quad x = 2\tan\theta, \quad \mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta, \\
&= \int \frac{1}{\sqrt{4+(2\tan\theta)^2}} \cdot 2\sec^2\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\sqrt{4(1+\tan^2\theta)}} \cdot 2\sec^2\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\sqrt{1+\tan^2\theta}} \sec^2\theta\,\mathrm{d}\theta \\
&= \int \frac{1}{\sqrt{\sec^2\theta}} \sec^2\theta\,\mathrm{d}\theta \\
&= \int \sec\theta\,\mathrm{d}\theta \\
&= \ln|\tan\theta+\sec\theta| + C = \boxed{\ln\left|\frac{x}{2}+\frac{\sqrt{4+x^2}}{2}\right| + C}
\end{aligned}$$