Math 152.02
Calculus with Analytic Geometry II

February 14, 2011

1. Lecture 17 - 2/14
   - Trig substitutions

Trig substitutions

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For integrals involving $\sqrt{a^2 - x^2}$ it may help to substitute

$$x = a\sin\theta$$

For integrals involving $\sqrt{x^2 + a^2}$ it may help to substitute

$$x = a\tan\theta$$

For integrals involving $\sqrt{x^2 - a^2}$ it may help to substitute

$$x = a\sec\theta$$

Note: Somtimes it may help to complete the square.

Example 106

Compute $\int \frac{t^3}{\sqrt{1-t^2}}\,\mathrm{d}t$

$$
\begin{aligned}
\int \frac{t^3}{\sqrt{1-t^2}}\,\mathrm{d}t & \qquad t = \sin\theta, \quad \mathrm{d}t = \cos\theta\,\mathrm{d}\theta, \\
&= \int \frac{\sin^3\theta}{\sqrt{1-(\sin\theta)^2}} \cdot \cos\theta\,\mathrm{d}\theta \\
&= \int \frac{\sin^3\theta}{\sqrt{\cos^2\theta}} \cdot \cos\theta\,\mathrm{d}\theta \\
&= \int \sin^3\theta\,\mathrm{d}\theta \\
& \qquad u = \cos\theta, \quad \mathrm{d}u = -\sin\theta\,\mathrm{d}\theta, \\
&= -\int \sin^2\theta \cdot (-\sin\theta)\,\mathrm{d}\theta \\
&= -\int (1-\cos^2\theta) \cdot (-\sin\theta)\,\mathrm{d}\theta \\
&= -\int 1-u^2\,\mathrm{d}u = -u + \frac{u^3}{3} + C \\
&= -\cos\theta + \frac{\cos^3\theta}{3} + C = \boxed{-\sqrt{1-t^2} + \frac{\left(\sqrt{1-t^2}\right)^3}{3} + C}
\end{aligned}
$$

Example 107

Compute $\int \frac{1}{\sqrt{4+x^2}}\, dx$

$$\int \frac{1}{\sqrt{4+x^2}}\, dx = \int \frac{1}{\sqrt{4+x^2}}\, dx$$

$$x = 2\tan\theta, \quad dx = 2\sec^2\theta\, d\theta,$$

$$= \int \frac{1}{\sqrt{4+(2\tan\theta)^2}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{4(1+\tan^2\theta)}} \cdot 2\sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{1+\tan^2\theta}} \sec^2\theta\, d\theta$$

$$= \int \frac{1}{\sqrt{\sec^2\theta}} \sec^2\theta\, d\theta$$

$$= \int \sec\theta\, d\theta$$

$$= \ln|\tan\theta + \sec\theta| + C = \boxed{\ln\left|\frac{x}{2} + \frac{\sqrt{4+x^2}}{2}\right| + C}$$

Example 108

Compute $\int \frac{1}{\left(\sqrt{x^2+2x+5}\right)^3}\,\mathrm{d}x$

$$\int \frac{1}{\left(\sqrt{x^2+2x+5}\right)^3}\,\mathrm{d}x = \int \frac{1}{\left(\sqrt{(x+1)^2+4}\right)^3}\,\mathrm{d}x$$

$$x+1 = 2\tan\theta, \quad \mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta,$$

$$= \int \frac{1}{\left(\sqrt{(2\tan\theta)^2+4}\right)^3}\cdot 2\sec^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int \frac{1}{\left(\sqrt{\tan^2\theta+1}\right)^3}\cdot \sec^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec^3\theta}\cdot \sec^2\theta\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec\theta}\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int \cos\theta\,\mathrm{d}\theta$$

$$= \frac{1}{4}\sin\theta + C = \boxed{\frac{x+1}{\sqrt{(x+1)^2+4}} + C}$$