# Practice Final Solutions - Math 1534

1. *(5 points)* Decide if the following statements are TRUE or FALSE and circle your answer. **You do NOT need to justify your answers.**

**T** (a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^3$ then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

**T** (b) If $\mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$ is a vector-valued function and $|\mathbf{r}(t)| = 4$ for all $t \in \mathbf{R}$ then $\mathbf{r} \cdot \mathbf{r}' = 0$ for all $t$.

**F** (c) Let $\mathbf{u}$ and $\mathbf{v}$ be a vectors in $\mathbf{R}^3$. Then $\mathbf{u} \cdot (\mathbf{u} - \mathbf{v})$ is a vector in $\mathbf{R}^3$.

**F** (d) Let $f = f(x, y)$ be a function of two variables which is continuous at $(a, b)$ and $g = g(t)$ be a function which is continuous at $f(a, b)$. Let

$$h(t) = f(g(t), g(t)).$$

Then $h$ is continuous at $f(a, b)$.

**T** (e) Let

$$\sum_{k=0}^{\infty} a_k (x - 3)^k$$

be a power series which converges absolutely at $x = 0$. Then the power series converges at $x = 6$.

2. *(5 points)* Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

(a) Give an example of a continuous vector-valued function $\mathbf{r}(t)$ which is *not* differentiable at $t = 2$.

$\boxed{\mathbf{r}(t) = |t - 2|\mathbf{i} + t\mathbf{j}}$

(b) Give an example of a vector-valued function which has constant curvature $\kappa \neq 0$.

$\boxed{\mathbf{r}(t) = \langle \cos t, \sin t \rangle}$

(c) Give equations for two different planes in $\mathbf{R}^3$ which are parallel to the plane $z = 2x - 5y$.

$\boxed{2x - 5y - z = 1}$ and $\boxed{2x - 5y - z = 2}$

(d) Give an example of a vector $\mathbf{u} \in \mathbf{R}^2$ for which $\mathbf{u} \cdot \left\langle \cos \frac{\pi}{5}, \sin \frac{\pi}{5} \right\rangle = 0$

$\boxed{\mathbf{u} = \mathbf{0} = \langle 0, 0 \rangle}$

(e) Give an example of a positive convergent series for which the ratio test is inconclusive.

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2}}$$

3. *(5 points)* Decide whether the series

$$\sum_{k=1}^{\infty} \ln\left(1+\frac{1}{k^2}\right)$$

converges absolutely, converges conditionally or diverges.

The Taylor series for $\ln(1+x)$ centered at $0$ is

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx \\ &= \int \frac{1}{1-(-x)}\,dx \\ &= \int 1+(-x)+(-x)^2+(-x)^3+\cdots\,dx \\ &= \int 1-x+x^2-x^3+\cdots\,dx \\ &= C+x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\end{aligned}$$

$\ln(1+0)=0$ so $C=0$. Thus if $|x|<1$

$$\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

If $\frac{1}{k^2}<1$ then

$$\begin{aligned}\ln\left(1+\frac{1}{k^2}\right) &= \left(\frac{1}{k^2}\right)-\frac{\left(\frac{1}{k^2}\right)^2}{2}+\frac{\left(\frac{1}{k^2}\right)^3}{3}-\frac{\left(\frac{1}{k^2}\right)^4}{4}\cdots \\ &= \frac{1}{k^2}-\frac{1}{2k^4}+\frac{1}{3k^6}-\frac{1}{4k^8}+\cdots\end{aligned}$$

We will use the limit comparison test to compare the series $\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2}\right)$ to the series $\sum_{k=1}^{\infty}\frac{1}{k^2}$.

$$\begin{aligned}\lim_{k\to\infty}\frac{\ln\left(1+\frac{1}{k^2}\right)}{\left(\frac{1}{k^2}\right)} &= \lim_{k\to\infty}\frac{\frac{1}{k^2}-\frac{1}{2k^4}+\frac{1}{3k^6}-\frac{1}{4k^8}+\cdots}{\left(\frac{1}{k^2}\right)} \\ &= \lim_{k\to\infty} 1-\frac{1}{2k^2}+\frac{1}{3k^4}-\frac{1}{4k^6}+\cdots \\ &= 1\neq 0\end{aligned}$$

The series $\sum_{k=1}^{\infty}\frac{1}{k^2}$ converges absolutely since it is a $p$-series with $p=2$ so by the limit comparison test $\boxed{\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2}\right) \text{ converges absolutely.}}$

4. *(3 points)* Give a function $f$ which satisfies

$$\begin{aligned}f(2) &= 2 \\ f'(2) &= 4 \\ f''(2) &= -3 \\ f'''(2) &= 10.\end{aligned}$$

$$\boxed{f(x) = 2+4(x-2)+\frac{-3(x-2)^2}{2!}+\frac{10(x-2)^3}{3!}}$$

5. *(5 points)* Let $\mathbf{u} = \langle 3, 2, 2\rangle$ and $\mathbf{v} = \langle 1, -4, 6\rangle$.

(a) Compute $2\mathbf{u} - \mathbf{v}$.

$2\mathbf{u} - \mathbf{v} = 2\langle 3,2,2\rangle - \langle 1,-4,6\rangle = \langle 6,4,4\rangle - \langle 1,-4,6\rangle = \boxed{\langle 5,8,-2\rangle}$

(b) Compute $\mathbf{u} \cdot \mathbf{v}$.

$\mathbf{u} \cdot \mathbf{v} = \langle 3,2,2\rangle \cdot \langle 1,-4,6\rangle = 3 \cdot 1 + 2(-4) + 2 \cdot 6 = \boxed{7}$

(c) Compute $\mathbf{u} \times \mathbf{v}$.

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 1 & -4 & 6 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 2 \\ -4 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} \mathbf{k} \\
&= (2 \cdot 6 - (-4) \cdot 2)\mathbf{i} - (3 \cdot 6 - 1 \cdot 2)\mathbf{j} + (3 \cdot (-4) - 1 \cdot 2)\mathbf{k} \\
&= \boxed{20\mathbf{i} - 16\mathbf{j} - 14\mathbf{k}}
\end{aligned}$$

(d) Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
\text{Angle between } \mathbf{u} \text{ and } \mathbf{v} &= \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) \\
&= \arccos\left(\frac{\langle 3,2,2\rangle \cdot \langle 1,-4,6\rangle}{|\langle 3,2,2\rangle|\,|\langle 1,-4,6\rangle|}\right) \\
&= \arccos\left(\frac{7}{\sqrt{3^2+2^2+2^2}\sqrt{1^2+(-4)^2+6^2}}\right) \\
&= \arccos\left(\frac{7}{\sqrt{17 \cdot 53}}\right) \\
&= \boxed{\arccos\left(\frac{7}{\sqrt{901}}\right)}
\end{aligned}$$

(e) Compute $\text{proj}_{\mathbf{u}}\mathbf{v}$

$$\begin{aligned}
\text{proj}_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \\
&= \frac{\langle 1,-4,6\rangle \cdot \langle 3,2,2\rangle}{\langle 3,2,2\rangle \cdot \langle 3,2,2\rangle}\langle 3,2,2\rangle \\
&= \frac{1 \cdot 3 - 4 \cdot 2 + 6 \cdot 2}{3 \cdot 3 + 2 \cdot 2 + 2 \cdot 2}\langle 3,2,2\rangle \\
&= \frac{7}{17}\langle 3,2,2\rangle \\
&= \boxed{\left\langle \frac{21}{17}, \frac{14}{17}, \frac{14}{17} \right\rangle}
\end{aligned}$$

6. *(5 points)* A ball is thrown from an intial height of 20 m above with an initial speed of 10 m/s. and initial angle of $\frac{\pi}{3}$ radians with the ground.

(a) Give the position vector of the ball $\mathbf{r}(t)$ at time $t$.

The acceleration of gravity is constant so at time $t$ the acceleration of the ball is

$$\mathbf{a}(t) = \langle 0, -g \rangle = \langle 0, -9.8 \rangle.$$

Let $\mathbf{v}_0$ be the initial velocity. Since the initial speed is 10 and the ball is thrown at angle $\frac{\pi}{3}$ we have

$$\mathbf{v}_0 = 10 \left\langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3} \right\rangle = 10 \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \left\langle 5, 5\sqrt{3} \right\rangle.$$

Thus at time $t$ the velocity of the ball is

$$\begin{aligned}
\mathbf{v}(t) &= \int_0^t \mathbf{a}(w)\, dw + \mathbf{v}_0 \\
&= \int_0^t \langle 0, -9.8 \rangle\, dw + \left\langle 5, 5\sqrt{3} \right\rangle \\
&= \langle 0, -9.8w \rangle \Big|_0^t + \left\langle 5, 5\sqrt{3} \right\rangle \\
&= \langle 0, -9.8t \rangle + \left\langle 5, 5\sqrt{3} \right\rangle \\
&= \langle 5, -9.8t + 5\sqrt{3} \rangle
\end{aligned}$$

Let $\mathbf{r}_0$ be the initial position. Since the ball starts at height 20 we have

$$\mathbf{r}_0 = \langle 0, 20 \rangle.$$

Thus at time $t$ the position of the ball is

$$\begin{aligned}
\mathbf{r}(t) &= \int_0^t \mathbf{v}(w)\, dw + \mathbf{r}_0 \\
&= \int_0^t \langle 5, -9.8w + 5\sqrt{3} \rangle\, dw + \langle 0, 20 \rangle \\
&= \left\langle 5w, \frac{-9.8}{2} w^2 + 5\sqrt{3}w \right\rangle \Bigg|_0^t + \langle 0, 20 \rangle \\
&= \left\langle 5t, \frac{-9.8}{2} t^2 + 5\sqrt{3}t \right\rangle + \langle 0, 20 \rangle \\
&= \boxed{\left\langle 5t, -4.9t^2 + 5\sqrt{3}t + 20 \right\rangle}
\end{aligned}$$

(b) Set up but do not evaluate an integral which gives the length of the path traveled by the ball. This is **not** the distance traveled by the ball along the ground, but the length of the path that the ball traces in the air.

First we need to calculate the amount of time $T$ that the ball is in the air. At time $T$ the ball hits the ground so the height will be $0$. Thus we need to solve for $T$ in the equation:

$$-4.9T^2 + 5\sqrt{3}T + 20 = 0$$

Using the quadratic equation we get that

$$\begin{aligned} T &= \frac{-5\sqrt{3} \pm \sqrt{(5\sqrt{3})^2 - 4(-4.9)20}}{2(-4.9)} \\ &= \frac{-5\sqrt{3} \pm \sqrt{25 \cdot 3 + 80 \cdot 4.9}}{-9.8} \\ &= \frac{5\sqrt{3} \pm \sqrt{75 + 8 \cdot 49}}{9.8} \end{aligned}$$

We want the positive root so

$$T = \frac{5\sqrt{3} + \sqrt{75 + 392}}{9.8} = \frac{\sqrt{75} + \sqrt{467}}{9.8}$$

Therefore the length of the path travelled by the ball is

$$\begin{aligned} L &= \int_0^T |\mathbf{v}(t)|\, dt \\ &= \int_0^T \left|\langle 5, -9.8t + 5\sqrt{3}\rangle\right| dt \\ &= \int_0^T \sqrt{5^2 + (-9.8t + 5\sqrt{3})^2}\, dt \\ &= \boxed{\int_0^{\frac{\sqrt{75}+\sqrt{467}}{9.8}} \sqrt{5^2 + (-9.8t + 5\sqrt{3})^2}\, dt} \end{aligned}$$

7. *(5 points)* A weight of 5 N is tied to two strings, both fastened to the ceiling. The first string is tied to the weight at position $\langle 0, 0\rangle$ and is fastened to the ceiling at position $\langle 3, 2\rangle$ The second string is tied to the weight at position $\langle 0, 0\rangle$ and is fastened to the ceiling at position $\langle -1, 2\rangle$. Compute the force vectors which give the forces that the strings exhert on the weight.

Let $\mathbf{F}_1$ be the force applied by the first string and $\mathbf{F}_2$ be the force applied by the second string. There must be scalars $a$ and $b$ such that

$$\begin{aligned} \mathbf{F}_1 &= a\big(\langle 3, 2\rangle - \langle 0, 0\rangle\big) = \langle 3a, 2a\rangle \\ \mathbf{F}_2 &= b\big(\langle -1, 2\rangle - \langle 0, 0\rangle\big) = \langle -b, 2b\rangle \end{aligned}$$

Let $\mathbf{G}$ be the force of gravity on the weight. The force has magnitude 5 N so

$$\mathbf{G} = \langle 0, -5\rangle$$

The weight is stationary so all of the forces must balance. Hence

$$\begin{aligned} &\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{G} = \mathbf{0} \\ \implies &\langle 3a, 2a\rangle + \langle -b, 2b\rangle + \langle 0, -5\rangle = \langle 0, 0\rangle \\ \implies &\langle 3a - b, 2a + 2b - 5\rangle = \langle 0, 0\rangle \\ \implies &3a - b = 0 \text{ and } 2a + 2b - 5 = 0 \end{aligned}$$

Solving for $b$ in terms of $a$ we get $b = 3a$. Now substituting in

$$\begin{aligned} &2a + 2(3a) - 5 = 0 \\ \implies &8a - 5 = 0 \\ \implies &8a = 5 \\ \implies &a = \tfrac{5}{8} \\ \implies &b = 3 \cdot \tfrac{5}{8} = \tfrac{15}{8}. \end{aligned}$$

Hence the force vector applied by the first string is

$$\mathbf{F}_1 = a\langle 3,2\rangle = \tfrac{5}{8}\langle 3,2\rangle = \boxed{\langle \tfrac{15}{8}, \tfrac{5}{4}\rangle}$$

The force vector applied by the second string is

$$\mathbf{F}_2 = b\langle -1,2\rangle = \tfrac{15}{8}\langle -1,2\rangle = \boxed{\langle \tfrac{-15}{8}, \tfrac{15}{4}\rangle}$$

8. *(5 points)* Compute the maximum and minimum curvature for the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Curvature is most easily calculated for vector-valued functions so we will parametrize the ellipse:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

If we let $\cos t = \frac{x}{2}$ and $\sin t = \frac{y}{3}$ then $x = 2\cos t$ and $y = 3\sin t$ so we get the parametrization

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle = \langle 2\cos t, 3\sin t\rangle \qquad t \in [0, 2\pi]$$

We can compute curvature at $t$ using the formula $\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}$ where $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{r}''$.

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) = \langle -2\sin t, 3\cos t\rangle \\ \mathbf{a}(t) &= \mathbf{r}''(t) = \langle -2\cos t, -3\sin t\rangle \end{aligned}$$

Thus

$$\begin{aligned} \mathbf{v}\times\mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} \\ &= \begin{vmatrix} 3\cos t & 0 \\ -3\sin t & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -2\sin t & 0 \\ -2\cos t & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -2\sin t & 3\cos t \\ -2\cos t & -3\sin t \end{vmatrix}\mathbf{k} \\ &= ((3\cos t)\cdot 0 - (-3\sin t)\cdot 0)\mathbf{i} - (-2\sin t\cdot 0 - (-2\cos t)\cdot 0)\mathbf{j} \\ &\quad + ((-2\sin t)(-3 sin t) - (-2\cos t)(3\cos t))\mathbf{k} \\ &= (6\sin^2 t + 6\cos^2 t)\mathbf{k} \\ &= 6\mathbf{k} \end{aligned}$$

Therefore

$$|\mathbf{v}\times\mathbf{a}| = |6\mathbf{k}| = \sqrt{0^2+0^2+6^2} = 6$$

We can also compute

$$\begin{aligned} |\mathbf{v}|^3 &= |\langle -2\sin t, 3\cos t\rangle|^3 \\ &= \left(\sqrt{(-2\sin t)^2 + (3\cos t)^2}\right)^3 \\ &= (4\sin^2 t + 9\cos^2 t)^{\frac{3}{2}} \end{aligned}$$

Thus at time $t$ the curvature is

$$\begin{aligned}\kappa(t) &= \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3}\\ &= \frac{6}{\left(4\sin^2 t + 9\cos^2 t\right)^{\frac{3}{2}}}\\ &= 6\left(4\sin^2 t + 9\cos^2 t\right)^{-\frac{3}{2}}\end{aligned}$$

We want the maximum and minimum curvatures so we need the critical values of the curvature function

$$\begin{aligned}\kappa'(t) &= 6\left(-\tfrac{3}{2}\right)\left(4\sin^2 t + 9\cos^2 t\right)^{-\frac{5}{2}}\left(4\cdot 2\sin t\cos t + 9\cdot 2\cos t(-\sin t)\right)\\ &= \frac{(-9)(-5)2\sin t\cos t}{\left(4\sin^2 t + 9\cos^2 t\right)^{\frac{5}{2}}}\\ &= \frac{45\sin 2t}{\left(4\sin^2 t + 9\cos^2 t\right)^{\frac{5}{2}}}\end{aligned}$$

Critical values for $\kappa(t)$ will occur where $45\sin 2t = 0$ or where $4\sin^2 t + 9\cos^2 t = 0$. Since $\sin t$ and $\cos t$ are never both $0$ for the same $t$ we have $4\sin^2 t + 9\cos^2 t \neq 0$ for all $t$.

Therefore we need to solve for $t$ in the equation

$$\begin{aligned}&45\sin 2t = 0\\ \Longrightarrow\ &\sin 2t = 0\\ \Longrightarrow\ &2t = n\pi\\ \Longrightarrow\ &t = \frac{n\pi}{2}\end{aligned}$$

The domain of our parametrization is $t \in [0, 2\pi]$ so we need to test the 5 critical values and endpoints: $t = 0, t = \frac{\pi}{2}, t = \pi, t = \frac{3\pi}{2}, t = 2\pi$

$$\begin{aligned}\kappa(0) &= 6\left(4\sin^2 0 + 9\cos^2 0\right)^{-\frac{3}{2}} = 6\cdot 9^{-\frac{3}{2}} = 6\cdot 3^{-3} = \frac{6}{27} = \frac{2}{9}\\ \kappa\left(\tfrac{\pi}{2}\right) &= 6\left(4\sin^2 \tfrac{\pi}{2} + 9\cos^2 \tfrac{\pi}{2}\right)^{-\frac{3}{2}} = 6\cdot 4^{-\frac{3}{2}} = 6\cdot 2^{-3} = \frac{6}{8} = \frac{3}{4}\\ \kappa(\pi) &= 6\left(4\sin^2 \pi + 9\cos^2 \pi\right)^{-\frac{3}{2}} = 6\cdot 9^{-\frac{3}{2}} = \frac{2}{9}\\ \kappa\left(\tfrac{3\pi}{2}\right) &= 6\left(4\sin^2 \tfrac{3\pi}{2} + 9\cos^2 \tfrac{3\pi}{2}\right)^{-\frac{3}{2}} = 6\cdot 4^{-\frac{3}{2}} = \frac{3}{4}\\ \kappa(2\pi) &= 6\left(4\sin^2 2\pi + 9\cos^2 2\pi\right)^{-\frac{3}{2}} = 6\cdot 9^{-\frac{3}{2}} = \frac{2}{9}\end{aligned}$$

Thus we can conclude the $\boxed{\text{the maximum curvature is } \frac{3}{4}}$ and $\boxed{\text{the minimum curvature is } \frac{2}{9}.}$

9. *(5 points)* Give an equation of the plane containing the line $\mathbf{r}(t) = \langle 3+t, 2, 1-t\rangle$ and the point $\langle 2, -3, 1\rangle$.

First we need two vectors parallel to the plane.

$$\begin{aligned}\mathbf{v}_1 &= \mathbf{r}(1) - \mathbf{r}(0) = \langle 3+1, 2, 1-1\rangle - \langle 3+0, 2, 1-0\rangle = \langle 1, 0, -1\rangle\\ \mathbf{v}_2 &= \langle 2, -3, 1\rangle - \mathbf{r}(0) = \langle 2, -3, 1\rangle - \langle 3+0, 2, 1-0\rangle = \langle -1, -5, 0\rangle\end{aligned}$$

A normal vector for the plane will be

$$\begin{aligned}
\mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -1 & -5 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ -1 & -5 \end{vmatrix} \mathbf{k} \\
&= (0 \cdot 0 - (-5)(-1))\mathbf{i} - (1 \cdot 0 - (-1)(-1))\mathbf{j} + (1(-5) - (-1) \cdot 0)\mathbf{k} \\
&= -5\mathbf{i} + \mathbf{j} - 5\mathbf{k}
\end{aligned}$$

Thus an equation for the plane is

$$\begin{gathered}
\mathbf{n} \cdot \langle x-2, y-(-3), z-1 \rangle = 0 \\
-5(x-2) + (y+3) - 5(z-1) = 0 \\
-5x + 10 + y + 3 - 5z + 5 = 0 \\
\boxed{-5x + y - 5z = -18}
\end{gathered}$$

10. *(5 points)* Compute the normal and tangential components of acceleration ($a_N$ and $a_T$) for a particle whose position at time $t$ is given by the vector-valued function

$$\mathbf{r}(t) = \langle t^2, t+2 \rangle.$$

$$\begin{aligned}
\mathbf{v}(t) &= \mathbf{r}'(t) = \langle 2t, 1 \rangle \\
\mathbf{a}(t) &= \mathbf{r}''(t) = \langle 2, 0 \rangle
\end{aligned}$$

$$\begin{aligned}
a_T &= |\text{proj}_{\mathbf{v}} \mathbf{a}| \\
&= \left| \frac{\mathbf{a} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \right| \\
&= \frac{|\mathbf{a} \cdot \mathbf{v}|}{|\mathbf{v}|} \\
&= \frac{|\langle 2, 0 \rangle \cdot \langle 2t, 1 \rangle|}{|\langle 2t, 1 \rangle|} \\
&= \boxed{\frac{|4t|}{\sqrt{4t^2 + 1}}}
\end{aligned}$$

$$
\begin{aligned}
a_N &= |a_N \mathbf{N}| \\
&= |\mathbf{a} - a_T \mathbf{T}| \\
&= \left| \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} \right| \\
&= \left| \langle 2,0\rangle - \frac{\langle 2,0\rangle\cdot\langle 2t,1\rangle}{\langle 2t,1\rangle\cdot\langle 2t,1\rangle}\langle 2t,1\rangle \right| \\
&= \left| \langle 2,0\rangle - \frac{4t}{4t^2+1}\langle 2t,1\rangle \right| \\
&= \left| \langle 2,0\rangle - \left\langle \frac{8t^2}{4t^2+1}, \frac{4t}{4t^2+1} \right\rangle \right| \\
&= \left| \left\langle \frac{8t^2+2}{4t^2+1} - \frac{8t^2}{4t^2+1}, \frac{4t}{4t^2+1} \right\rangle \right| \\
&= \left| \left\langle \frac{2}{4t^2+1}, \frac{4t}{4t^2+1} \right\rangle \right| \\
&= \sqrt{\left(\frac{2}{4t^2+1}\right)^2 + \left(\frac{4t}{4t^2+1}\right)^2} \\
&= \sqrt{\frac{4+16t^2}{(4t^2+1)^2}} \\
&= \frac{\sqrt{4+16t^2}}{4t^2+1} \\
&= \frac{2\sqrt{1+4t^2}}{4t^2+1} \\
&= \boxed{\frac{2}{\sqrt{4t^2+1}}}
\end{aligned}
$$

11. *(3 points)* Let $f(x,y) = y^2 e^{xy} - x^3 y$

(a) Compute the partial derivative $f_x$.

$$f_x = y^2 e^{xy} \cdot y - 3x^2 y = \boxed{y^3 e^{xy} - 3x^2 y}$$

(b) Compute $\frac{\partial^2 f}{\partial y \partial x}$.

$$\frac{\partial^2 f}{\partial y \partial x} = 3y^2 e^{xy} + y^3 e^{xy} \cdot x - 3x^2 = \boxed{3y^2 e^{xy} + xy^3 e^{xy} - 3x^2}$$

(c) Compute $\frac{\partial^2 f}{\partial x \partial y}$.

By equality of mixed partials,

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \boxed{3y^2 e^{xy} + xy^3 e^{xy} - 3x^2}$$

12. *(5 points)* Give a Taylor polynomial for $f(x) = \cos x$ which estimates $\cos x$ to within $\frac{1}{100}$ for all $x \in [5\pi, 6\pi]$.

Since the center for the Taylor polynomial is not specified we will choose to compute the Taylor polynomial for $f(x) = \cos x$ centered at $x = \frac{6\pi+5\pi}{2} = \frac{11\pi}{2}$.

$$\begin{aligned} f(x) &= \cos x & f(\tfrac{11\pi}{2}) &= \cos\tfrac{11\pi}{2} = 0 \\ f'(x) &= -\sin x & f'(\tfrac{11\pi}{2}) &= -\sin\tfrac{11\pi}{2} = 1 \\ f''(x) &= -\cos x & f''(\tfrac{11\pi}{2}) &= -\cos\tfrac{11\pi}{2} = 0 \\ f'''(x) &= \sin x & f'''(\tfrac{11\pi}{2}) &= \sin\tfrac{11\pi}{2} = -1 \\ f^{(4)}(x) &= \cos x & f^{(4)}(\tfrac{11\pi}{2}) &= \cos\tfrac{11\pi}{2} = 0 \\ &\vdots & &\vdots \end{aligned}$$

So the $(2n+1)$st Taylor polynomial for $f(x) = \cos x$ centered at $\frac{11\pi}{2}$ is

$$P_{2n+1}(x) = \sum_{k=0}^{n} \frac{(x - \frac{11\pi}{2})^{2k+1}}{(2k+1)!}$$

By Taylor's Theorem for each $x$ there is a number $c$ between $\frac{11\pi}{2}$ and $x$ such that the remainder for this polynomial will be

$$R_{2n+1}(x) = \frac{f^{(n+2)}(c)}{(2n+2)!}\left(x - \frac{11\pi}{2}\right)^{2n+2}$$

Notice that for all $n$ and all $c$ we have $f^{(n+2)}(c) = \pm\sin(c)$ or $f^{(n+2)}(c) = \pm\cos(c)$ so $|f^{(n+2)}(c)| \le 1$. Therefore

$$\begin{aligned} & n = 3 \\ \implies\ & 2n = 6 \\ \implies\ & 2n+2 = 8 \\ \implies\ & \frac{(2n+2)!}{2^{2n+2}} > 100 \quad \text{NOTE: I needed a calculator here!} \\ \implies\ & \frac{2^{2n+2}}{(2n+2)!} < \frac{1}{100} \\ \implies\ & \frac{\left(\frac{\pi}{2}\right)^{2n+1}}{(2n+2)!} < \frac{1}{100} \\ \implies\ & \left|\frac{1}{(2n+2)!}(x - \tfrac{11\pi}{2})^{2n+2}\right| < \frac{1}{100} \qquad x \in [5\pi, 6\pi] \\ \implies\ & \left|\frac{f^{(n+1)}(c)}{(2n+2)!}(x - \tfrac{11\pi}{2})^{2n+2}\right| < \frac{1}{100} \qquad x, c \in [5\pi, 6\pi] \\ \implies\ & |R_{2n+1}(x)| < \frac{1}{100}. \end{aligned}$$

Thus

$$\begin{aligned} P_{2\cdot 3+1}(x) &= \sum_{k=0}^{3} \frac{(x - \frac{11\pi}{2})^{2k+1}}{(2k+1)!} \\ &= \boxed{(x - \tfrac{11\pi}{2}) - \frac{(x - \frac{11\pi}{2})^3}{3!} + \frac{(x - \frac{11\pi}{2})^5}{5!} - \frac{(x - \frac{11\pi}{2})^7}{7!}} \end{aligned}$$

will estimate $\cos x$ to within $\frac{1}{100}$ for all $x \in [5\pi, 6\pi]$.