Practice Midterm Solutions - Math 1534

1. *(5 points)* Decide if the following statements are TRUE or FALSE and circle your answer. **You do NOT need to justify your answers.**

   F (a) If $\sum_{k=0}^{\infty} a_k$ is a convergent series then $\sum_{k=0}^{\infty} a_{2k}$ is a convergent series.

   F (b) If the power series $\sum_{n=0}^{\infty} c_n(x-2)^n$ converges at $x=4$ then it must converge at $x=-1$.

   T (c) If the radius of convergence of the power series $\sum_{n=0}^{\infty} b_n x^n$ is 0 then the series $\sum_{n=0}^{\infty} b_n$ diverges.

   T (d) For any $r \in \mathbf{R}$ and $\theta \in \mathbf{R}$ the points (r,θ) and $(-r,\theta+\pi)$ represent the same point in polar coordinates.

   F (e) If the Taylor series for f at 0 is

   $$f(x) = \sum_{k=0}^{\infty} a_n x^n$$

   and has radius of convergence 1 then $f^{(4)}(0) = \frac{a_4}{4!}$.

2. *(5 points)* Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

   (a) Give two **different** ways to represent the point $(x,y)=(-1,-1)$ in polar coordinates.

   $(r,\theta) = (\sqrt{2}, \frac{5\pi}{4})$ and $(r,\theta) = (\sqrt{2}, -\frac{3\pi}{4})$

   (b) Give an example of a power series centered at 3 which has radius of convergence 0.

   $\sum_{k=0}^{\infty} k!(x-3)^k$

   (c) Give an example of a p-series that diverges.

   $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}}$

   (d) Give two **different** parametrizations of the line $y = 3x+2$.

   $x=t, y=3t+2$ and $x=t+2, y=3(t+2)+2$

   (e) Give an example of a sequence that grows faster than $\{c^n\}_{n=1}^{\infty}$ for any $c>0$.

   $\{n!\}_{n=1}^{\infty}$

3. *(15 points)* Show that the following series converge absolutely, converge conditionally or diverge:

   (a) $\sum_{k=2}^{\infty} \frac{2}{k(\ln k)^4}$

   The sequence $\left\{\frac{2}{k(\ln k)^4}\right\}_{k=2}^{\infty}$ is positive and decreasing so by integral test the series $\sum_{k=2}^{\infty} \frac{2}{k(\ln k)^4}$ will converge if and only if the improper integral $\int_2^{\infty} \frac{2}{x(\ln x)^4}\,dx$ converges.

   $$\begin{aligned}
   \int_2^{\infty} \frac{2}{x(\ln x)^4}\,dx &= \lim_{b\to\infty} \int_2^b \frac{2}{x(\ln x)^4}\,dx \\
   &\qquad (\text{Let } u = \ln x,\ du = \tfrac{dx}{x}) \\
   &= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{2}{u^4}\,du \\
   &= \lim_{b\to\infty} \left.\frac{2}{-3u^3}\right|_{u=\ln 2}^{\ln b} \\
   &= \lim_{b\to\infty} \frac{2}{-3(\ln b)^3} - \frac{2}{-3(\ln 2)^3} \\
   &= \frac{2}{3(\ln 2)^3}
   \end{aligned}$$

   This integral converges. Thus, by the integral test $\sum_{k=2}^{\infty} \frac{2}{k(\ln k)^4}$ converges. The series is positive so, $\sum_{k=2}^{\infty} \frac{2}{k(\ln k)^4}$ converges absolutely.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

The sequence $\left\{\frac{1}{\ln n}\right\}_{n=2}^{\infty}$ is positive and decreasing and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ so by the alternating series test the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges.

$\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{\ln n}\right| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$. For all $n > 0$

$$\ln n < n$$
$$\Rightarrow \frac{1}{\ln n} > \frac{1}{n}$$

$\sum_{n=2}^{\infty} \frac{1}{n}$ diverges since it is a p-series with $p = 1$. Thus by the comparison test $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

Thus we can conclude that $\boxed{\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \text{ converges conditionally.}}$

(c) $\sum_{k=1}^{\infty} \sin \frac{1}{k} \sec^2 \frac{1}{k}$

Let $a_k = \sin \frac{1}{k} \sec^2 \frac{1}{k}$ and $b_k = \frac{1}{k}$. We will use the limit comparison test to compare $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$.

$$\begin{aligned}
\lim_{k\to\infty} \frac{a_k}{b_k} &= \lim_{k\to\infty} \frac{\sin \frac{1}{k} \sec^2 \frac{1}{k}}{\left(\frac{1}{k}\right)} \\
&= \lim_{k\to\infty} \sec^2 \frac{1}{k} \lim_{k\to\infty} \frac{\sin \frac{1}{k}}{\left(\frac{1}{k}\right)} \\
&= \lim_{k\to\infty} \frac{1}{\cos^2 \frac{1}{k}} \lim_{k\to\infty} \frac{\left(\frac{-1}{k^2}\right) \cos \frac{1}{k}}{\left(\frac{-1}{k^2}\right)} \\
&= \lim_{k\to\infty} \frac{1}{\cos^2 \frac{1}{k}} \lim_{k\to\infty} \cos \frac{1}{k} \\
&= \frac{1}{1^2} \cdot 1 = 1
\end{aligned}$$

$\sum_{k=1}^{\infty} \frac{1}{k}$ diverges since it is a p-series with $p = 1$. Thus by the limit comparison test $\boxed{\sum_{k=1}^{\infty} \sin \frac{1}{k} \sec^2 \frac{1}{k} \text{ diverges.}}$

4. *(10 points)* For the following power series compute the radius of convergence:

(a) $\sum_{k=1}^{\infty} k^2 x^k$

Let $a_k = k^2 x^k$. We will use the ratio test to compute the radius of convergence for $\sum_{k=1}^{\infty} a_k$.

$$\begin{aligned}
\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\left|(k+1)^2 x^{k+1}\right|}{|k^2 x^k|} \\
&= \lim_{k\to\infty} \left(\frac{k+1}{k}\right)^2 |x| \\
&= \lim_{k\to\infty} \frac{\left(\frac{1}{k^2}\right)}{\left(\frac{1}{k^2}\right)} \left(\frac{k+1}{k}\right)^2 |x| \\
&= \lim_{k\to\infty} \left(\frac{1 + \frac{1}{k}}{1}\right)^2 |x| \\
&= |x|
\end{aligned}$$

Thus the power series converges if $|x| < 1$ and diverges if $|x| > 1$. $\boxed{\text{The radius of convergence is } 1.}$

(b) $\sum_{n=2}^{\infty} \frac{nx^n}{\ln n}$

Let $a_n = \frac{nx^n}{\ln n}$. We will use the ratio test to compute the radius of convergence for $\sum_{n=2}^{\infty} a_n$.

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \frac{\left|\frac{(n+1)x^{n+1}}{\ln(n+1)}\right|}{\left|\frac{nx^n}{\ln n}\right|} \\
&= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{\ln n}{\ln(n+1)} \cdot |x| \\
&= \lim_{n\to\infty} \frac{n+1}{n} \cdot \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \cdot \lim_{n\to\infty} |x| \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1} \cdot \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n+1}\right)} \cdot \lim_{n\to\infty} |x| \\
&= \frac{1+0}{1} \cdot \lim_{n\to\infty} \frac{n+1}{n} \cdot |x| \\
&= |x|
\end{aligned}$$

Thus the power series converges if $|x| < 1$ and diverges if $|x| > 1$. The radius of convergence is 1.

5. *(5 points)* Give the interval of convergence of the power series

$$\sum_{k=1}^{\infty} \frac{(-1)^k (x+2)^k}{k^2}.$$

Let $a_k = \frac{(-1)^k (x+2)^k}{k^2}$. First we will use the ratio test to compute the radius of convergence for $\sum_{k=1}^{\infty} a_k$.

$$\begin{aligned}
\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} &= \lim_{k\to\infty} \frac{\left|\frac{(-1)^{k+1}(x+2)^{k+1}}{(k+1)^2}\right|}{\left|\frac{(-1)^k (x+2)^k}{k^2}\right|} \\
&= \lim_{k\to\infty} \left(\frac{k}{k+1}\right)^2 \cdot |x+2| \\
&= |x+2|
\end{aligned}$$

Thus the power series converges if $|x+2| < 1$ and diverges if $|x+2| > 1$. The radius of convergence is 1 and the power series is centered at -2.

Now we must check convergence at the endpoints. If $x = -2+1 = -1$ then

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^k (x+2)^k}{k^2} &= \sum_{k=1}^{\infty} \frac{(-1)^k (-1+2)^k}{k^2} \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}
\end{aligned}$$

which converges by the alternating series test.

If $x = -2-1 = -3$ then

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{(-1)^k (x+2)^k}{k^2} &= \sum_{k=1}^{\infty} \frac{(-1)^k (-3+2)^k}{k^2} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2}
\end{aligned}$$

which converges since it is a p-series with $p = 2$.

Thus the interval of convergence is $[-3, -1]$.

6. *(5 points)* Give a power series with center 0 for a solution to the differential equation $y' = y + 1$ satisfying the initial condition $y(0) = 1$.

Let $y = \sum_{n=0}^{\infty} a_n x^n$. Then $y(0) = 1$ implies $1 = \sum_{n=0}^{\infty} a_n 0^n = a_0$. Hence $a_0 = 1$.

$$\begin{aligned} y' &= \frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \frac{d}{dx} a_n x^n \\ &= \sum_{n=0}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ &= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \end{aligned}$$

$y' = y + 1$ so

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 1 + \sum_{n=0}^{\infty} a_n x^n \\ \Rightarrow &\sum_{n=0}^{\infty} ((n+1) a_{n+1} - a_n) x^n = 1 \end{aligned}$$

If $n = 0$ this gives us the equation $((0+1)a_{0+1} - a_0) = 1$ which implies that $a_1 = a_0 + 1 = 2$.

If $n > 0$ we get the recursive equation $(n+1)a_{n+1} - a_n = 0$ which implies that

$$a_{n+1} = \frac{a_n}{n+1}.$$

Hence we get $a_2 = \frac{a_1}{2} = \frac{2}{2}$. $a_3 = \frac{a_2}{3} = \frac{2}{2\cdot 3}$. $a_4 = \frac{a_3}{4} = \frac{2}{2\cdot 3\cdot 4}$. *etc.*
For $n > 0$ we get $a_n = \frac{2}{n!}$ so

$$\boxed{y = 1 + \sum_{n=1}^{\infty} \frac{2x^n}{n!}}$$

7. Find the equation for the tangent line to the parametric curve

$$x = \sin t, \quad y = e^t$$

at $t = 0$.

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}\sin t = \cos t \\ \frac{dy}{dt} &= \frac{d}{dt}e^t = e^t \end{aligned}$$

Thus

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t}{\cos t}$$

Therefore, the slope of tangent line at $t = 0$ is $\frac{e^0}{\cos 0} = 1$. When $t = 0$ we get the point $(\sin 0, e^0) = (0, 1)$.
Thus the tangent line has equation $y = 1(x - 0) + 1$.

$$\boxed{y = x + 1.}$$

8. Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(a) Give the formula for this ellipse in polar coordinates

To convert from cartesian coordinates to polar coordinates we need the equations $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} &\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ \implies &\frac{(r\cos\theta)^2}{a^2} + \frac{(r\sin\theta)^2}{b^2} = 1 \end{aligned}$$

Now solve for r in terms of θ.

$$\begin{aligned} &\implies r^2\frac{(\cos\theta)^2}{a^2} + r^2\frac{(\sin\theta)^2}{b^2} = 1 \\ &\implies r^2\left[\frac{(\cos\theta)^2}{a^2} + \frac{(\sin\theta)^2}{b^2}\right] = 1 \\ &\implies r^2 = \left[\frac{(\cos\theta)^2}{a^2} + \frac{(\sin\theta)^2}{b^2}\right]^{-1} \\ &\implies \boxed{r(\theta) = \frac{1}{\sqrt{\frac{(\cos\theta)^2}{a^2} + \frac{(\sin\theta)^2}{b^2}}}, \qquad \theta \in [0, 2\pi)} \end{aligned}$$

(b) Derive the area for this ellipse using the polar formula from part (a)

Area is given by the formula:

$$\begin{aligned} \text{Area} &= \int_0^{2\pi} \frac{1}{2}[r(\theta)]^2\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2} \cdot \left(\frac{1}{\sqrt{\frac{(\cos\theta)^2}{a^2} + \frac{(\sin\theta)^2}{b^2}}}\right)^2 d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \frac{1}{\frac{(\cos\theta)^2}{a^2} + \frac{(\sin\theta)^2}{b^2}}\, d\theta \\ &= \frac{1}{2}\int_0^{2\pi} \frac{a^2b^2}{b^2\cos^2\theta + a^2\sin^2\theta}\, d\theta \end{aligned}$$

This integral more difficult than I realized so you can stop here.

9. Give a partial sum which estimates the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}$$

to within $\frac{1}{1000}$. Be sure to justify your answer.

Let $a_k = \frac{(-1)^k}{\sqrt[3]{k}}$. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}$ is an alternating series so the nth remainder R_n satisfies $|R_n| \le |a_{n+1}|$.

$$\begin{aligned} n &= 999,999,999 \\ \implies n &= 1,000,000,000 - 1 \\ \implies n &\ge 1000^3 - 1 \\ \implies n+1 &\ge 1000^3 \\ \implies \sqrt[3]{n+1} &\ge 1000 \\ \implies \frac{1}{\sqrt[3]{n+1}} &\le \frac{1}{1000} \\ \implies \left|\frac{(-1)^{n+1}}{\sqrt[3]{n+1}}\right| &\le \frac{1}{1000} \\ \implies |a_{n+1}| &\le \frac{1}{1000} \\ \implies |R_n| &\le \frac{1}{1000}. \end{aligned}$$

Thus the partial sum $\boxed{\sum_{k=1}^{999,999,999} \frac{(-1)^k}{\sqrt[3]{k}}}$ is within $\frac{1}{1000}$ of $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt[3]{k}}$.

10. Give a Taylor polynomial centered at 0 which estimates

$$f(x) = \ln(1-x)$$

to within $\frac{1}{10}$ for all $x \in [0, \frac{1}{2}]$ Be sure to justify your answer.

The Taylor polynomial for $\frac{1}{1-x}$ centered at $x = 0$ is $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$.

$$\begin{aligned} f(x) &= \ln(1-x) \\ f'(x) &= (-1)(1-x)^{-1} \\ f''(x) &= (-1)\cdot 1(1-x)^{-2} \\ f'''(x) &= (-1)\cdot 1 \cdot 2(1-x)^{-3} \\ &\vdots \\ f^{(n)}(x) &= (-1)(n-1)!(1-x)^{-n} \end{aligned}$$

By Taylor's Theorem there is some c between x and 0 such that

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$$

$$\begin{aligned} n &= 9 \\ \implies n+1 &\ge 10 \\ \implies \frac{1}{n+1} &\le \frac{1}{10} \\ \implies \frac{(\frac{1}{2})^{n+1}}{(n+1)(\frac{1}{2})^{n+1}} &\le \frac{1}{10} \\ \implies \frac{x^{n+1}}{(n+1)(1-c)^{n+1}} &\le \frac{1}{10} \qquad \text{where } x, c \in [0, \tfrac{1}{2}] \\ \implies \left|\frac{(-1)((n+1)-1)!(1-c)^{-(n+1)}}{(n+1)!}x^{n+1}\right| &\le \frac{1}{10} \qquad \text{where } x, c \in [0, \tfrac{1}{2}] \\ \implies \left|\frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}\right| &\le \frac{1}{10} \qquad \text{where } x, c \in [0, \tfrac{1}{2}] \\ \implies |R_n(x)| &\le \frac{1}{10} \end{aligned}$$

Thus the 9th Taylor polynomial suffices. The constant term will be $f(0) = \ln(1-0) = 0$.

$$f^{(k)}(0) = (-1)(k-1)!(1-0)^{-k} = -(k-1)! \quad \text{for } k > 0$$

So

$$\begin{aligned} P_9(x) &= \sum_{k=1}^{9} \frac{f^{(k)}(0)}{k!} x^k \\ &= \sum_{k=1}^{9} \frac{-(k-1)!}{k!} x^k \\ &= \boxed{\sum_{k=1}^{9} \frac{-x^k}{k}} \end{aligned}$$