Practice Midterm - Math 1534

1. (5 points) Decide if the following statements are TRUE or FALSE and circle your answer. You do NOT need to justify your answers.

   T  F  (a) If \( \sum_{k=0}^{\infty} a_k \) is a convergent series then \( \sum_{k=0}^{\infty} a_{2k} \) is a convergent series.

   T  F  (b) If the power series \( \sum_{n=0}^{\infty} c_n(x-2)^n \) converges at \( x = 4 \) then it must converge at \( x = -1 \).

   T  F  (c) If the radius of convergence of the power series \( \sum_{n=0}^{\infty} b_n x^n \) is \( 0 \) then the series \( \sum_{n=0}^{\infty} b_n \) diverges.

   T  F  (d) For any \( r \in \mathbb{R} \) and \( \theta \in \mathbb{R} \) the points \((r, \theta)\) and \((-r, \theta + \pi)\) represent the same point in polar coordinates.

   T  F  (e) If the Taylor series for \( f \) at \( 0 \) is

   \[ f(x) = \sum_{k=0}^{\infty} a_n x^n \]

   and has radius of convergence \( 1 \) then \( f^{(4)}(0) = \frac{a_4}{4!} \).

2. (5 points) Give examples of the following. Be as explicit as possible. You do NOT need to justify your answers.

   (a) Give two different ways to represent the point \((x, y) = (-1, -1)\) in polar coordinates.

   (b) Give an example of a power series centered at \( 3 \) which has radius of convergence \( 0 \).

   (c) Give an example of a \( p \)-series that diverges.

   (d) Give two different parametrizations of the line \( y = 3x + 2 \).

   (e) Give an example of a sequence that grows faster than \( \{c^n\}_{n=1}^{\infty} \) for any \( c > 0 \).

3. (15 points) Show that the following series converge absolutely, converge conditionally or diverge:

   (a) \( \sum_{k=2}^{\infty} \frac{2}{k[\ln k]^2} \)

   (b) \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} \)

   (c) \( \sum_{k=1}^{\infty} \sin\left(\frac{1}{k}\right)\sec^2\left(\frac{1}{k}\right) \)

4. (10 points) For the following power series compute the radius of convergence:

   (a) \( \sum_{k=1}^{\infty} k^2 x^k \)

   (b) \( \sum_{n=2}^{\infty} \frac{n x^n}{\ln n} \)

5. (5 points) Give the interval of convergence of the power series

   \[ \sum_{k=1}^{\infty} \frac{(-1)^k (x+2)^k}{k^2}. \]

6. (5 points) Give a power series with center \( 0 \) for a solution to the differential equation \( y' = y + 1 \) satisfying the initial condition \( y(0) = 1 \).

7. Find the equation for the tangent line to the parametric curve

   \[ x = \sin t, \quad y = e^t \]

   at \( t = 0 \).
8. Consider the ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

(a) Give the formula for this ellipse in polar coordinates
(b) Derive the area for this ellipse using the polar formula from part (a)

9. Give a partial sum which estimates the series

\[ \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \]

to within \( \frac{1}{1000} \). Be sure to justify your answer.

10. Give a Taylor polynomial centered at 0 which estimates

\[ f(x) = \ln(1 - x) \]

to within \( \frac{1}{10} \) for all \( x \in [0, \frac{1}{2}] \) Be sure to justify your answer.