Math 161.01 - Accelerated Calculus I
108 min December 7, 2010

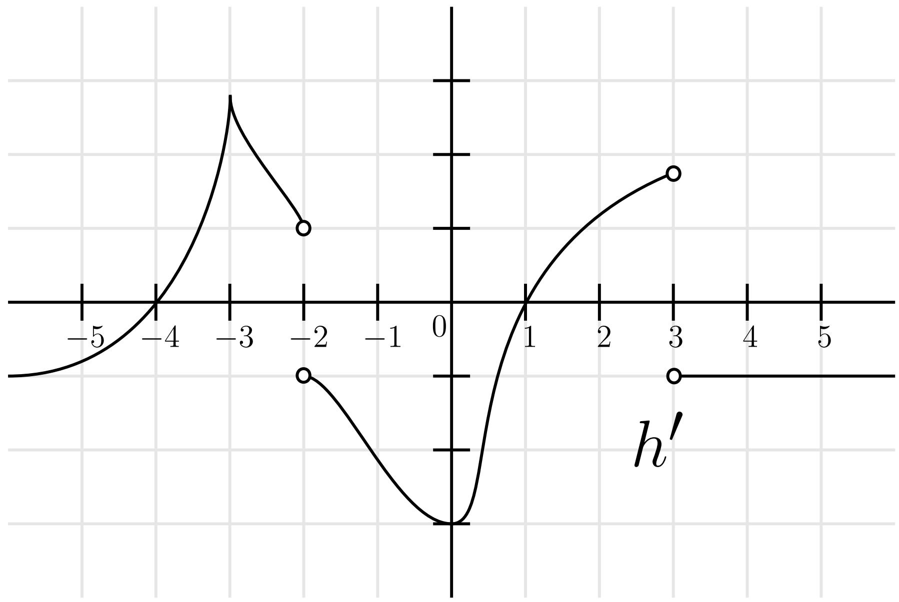
Practice Final

- Calculators are allowed as long as they have no symbolic integration capbility (TI-84 and comparable are ok)
- Remember to **CIRCLE YOUR FINAL ANSWER.**

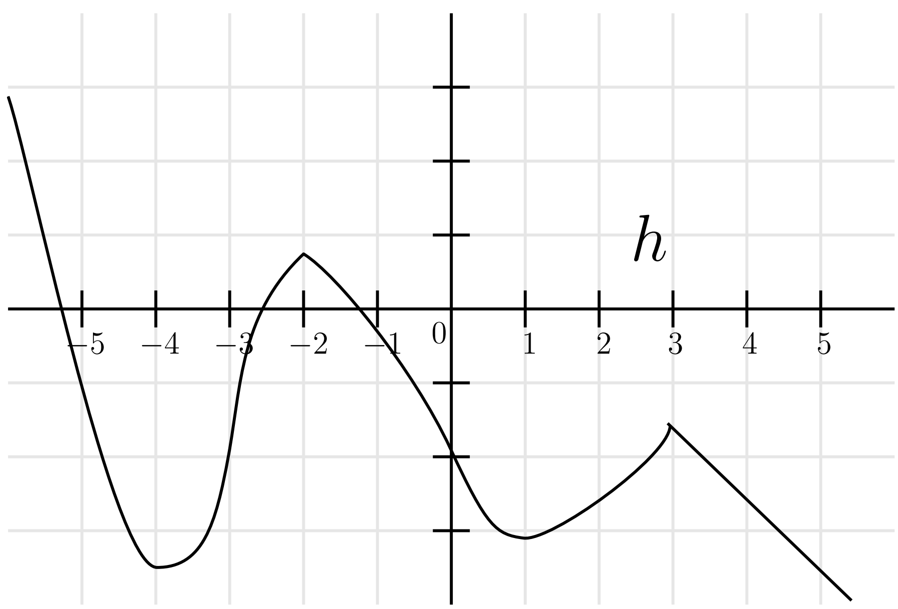
Useful facts:

- Volume of a sphere $V = \frac{4}{3}\pi r^3$
- Volume of a cone $V = \frac{1}{3}\pi r^2 h$
- $F = ma$ (force is mass times acceleration)
- $m = Vd$ (mass is volume times density)
- acceleration of gravity is $-9.8\,\mathrm{m/s^2}$
- $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

1. *(10 points)* The **derivative** of a **continuous** function h is pictured below.



Sketch a **continuous** function h whose derivative could be the given graph for h'.



2. Evaluate the following limits using any technique you like.

(a) $\lim_{x\to\infty} \sqrt{x+2} - \sqrt{x}$

$$\begin{aligned}
\lim_{x\to\infty} \sqrt{x+2} - \sqrt{x} &= \lim_{x\to\infty} (\sqrt{x+2} - \sqrt{x}) \cdot \frac{\sqrt{x+2}+\sqrt{x}}{\sqrt{x+2}+\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{x+2-x}{\sqrt{x+2}+\sqrt{x}} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{x+2}+\sqrt{x}} \\
&= \boxed{0}
\end{aligned}$$

(b) $\lim_{x\to\infty} \frac{(\ln x)^2}{\ln(\ln x)}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\ln x)^2}{\ln(\ln x)} &= \lim_{x\to\infty} \frac{2(\ln x)x^{-1}}{(\ln x)^{-1}x^{-1}} \\
&= \lim_{x\to\infty} 2(\ln x)^2 \\
&= \boxed{\infty}
\end{aligned}$$

(c) $\lim_{x\to\infty} \left(1+e^{-x}\right)^{\frac{1}{x}}$

$$\begin{aligned}
\lim_{x\to\infty} \left(1+e^{-x}\right)^{\frac{1}{x}} &= \lim_{x\to\infty} e^{x^{-1}\ln\left(1+e^{-x}\right)} \\
&= e^{\lim_{x\to\infty} x^{-1}\ln\left(1+e^{-x}\right)} \\
&= e^{0\cdot\ln 1} \\
&= \boxed{1}
\end{aligned}$$

3. Evaluate the following derivatives.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x + x^e + \arctan(3-8\pi^4)\right)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x + x^e + \arctan(3-8\pi^4)\right) = \boxed{e^x + ex^{(e-1)}}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x \sin(x^2)\right)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(e^x \sin(x^2)\right) = \boxed{e^x\sin(x^2) + 2xe^x\cos(x^2)}$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x+1}\right)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x+1}\right) = \frac{x+1-x}{(x+1)^2} = \boxed{\frac{1}{(x+1)^2}}$$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^x}{(x+1)^3\sin x}\right)$

Let $y = \frac{x^x}{(x+1)^3\sin x}$. Then

$$\ln y = x\ln x - 3\ln(x+1) - \ln(\sin x)$$

Hence,

$$\frac{y'}{y} = \ln x - 1 - \frac{3}{x+1} - \frac{\cos x}{\sin x}.$$

Hence,

$$y' = \boxed{\left(\frac{x^x}{(x+1)^3\sin x}\right)\left(\ln x - 1 - \frac{3}{x+1} - \cot x\right)}.$$

4. Evaluate the following indefinite integrals.

(a) $\displaystyle\int x\arctan x\,\mathrm{d}x$

$$u{=}\arctan x,\ \ \mathrm{d}u{=}\tfrac{\mathrm{d}x}{x^2+1},\ \ \mathrm{d}v{=}x\,\mathrm{d}x,\ \ v{=}\tfrac{x^2}{2}$$

$$\begin{aligned}\int x\arctan x\,\mathrm{d}x &= \frac{x^2\arctan x}{2} - \frac{1}{2}\int \frac{x^2}{x^2+1}\,\mathrm{d}x\\ &= \frac{x^2\arctan x}{2} - \frac{1}{2}\int 1 - \frac{1}{x^2+1}\,\mathrm{d}x\\ &= \boxed{\frac{x^2\arctan x}{2} - \frac{x}{2} + \frac{\arctan x}{2} + C}\end{aligned}$$

(b) $\displaystyle\int \sin^{17}x\cos^3 x\,\mathrm{d}x$

$$u{=}\sin x,\ \ \mathrm{d}u{=}\cos x\,\mathrm{d}x$$

$$\begin{aligned}\int \sin^{17}x\cos^3 x\,\mathrm{d}x &= \int \sin^{17}x(1-\sin^2 x)\cos x\,\mathrm{d}x\\ &= \int u^{17}(1-u^2)\,\mathrm{d}u\\ &= \int u^{17} - u^{19}\,\mathrm{d}u\\ &= \frac{u^{18}}{18} - \frac{u^{20}}{20} + C\\ &= \boxed{\frac{\sin^{18}x}{18} - \frac{\sin^{20}x}{20} + C}\end{aligned}$$

(c) $\int \frac{\sec^3 x}{\tan x}\,\mathrm{d}x$

$$\begin{aligned}
\int \frac{\sec^3 x}{\tan x}\,\mathrm{d}x &= \int \frac{\sec x(1+\tan^2 x)}{\tan x}\,\mathrm{d}x \\
&= \int \frac{\sec x}{\tan x}\,\mathrm{d}x + \int \frac{\sec x \tan^2 x}{\tan x}\,\mathrm{d}x \\
&= \int \csc x\,\mathrm{d}x + \int \sec x \tan x\,\mathrm{d}x \\
&= \boxed{\ln|\csc x - \cot x| + \sec x + C}
\end{aligned}$$

(d) $\int \sin^4 x\,\mathrm{d}x$

$$\begin{aligned}
\int \sin^4 x\,\mathrm{d}x &= \int \frac{(1-\cos 2x)^2}{2^2}\,\mathrm{d}x \\
&= \frac{1}{4}\int 1 - 2\cos 2x + \cos^2 2x\,\mathrm{d}x \\
&= \frac{1}{4}\int 1 - 2\cos 2x + \frac{1+\cos 4x}{2}\,\mathrm{d}x \\
&= \frac{1}{4}\int \frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}\,\mathrm{d}x \\
&= \frac{1}{4}\left(\frac{3x}{2} - \sin 2x + \frac{\sin 4x}{8}\right) + C \\
&= \boxed{\frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C}
\end{aligned}$$

(e) $\int \cos 2x \cos 9x\,\mathrm{d}x$

$$\begin{aligned}
\int \cos 2x \cos 9x\,\mathrm{d}x &= \frac{1}{2}\int \cos(2x-9x) + \cos(2x+9x)\,\mathrm{d}x \\
&= \frac{1}{2}\int \cos(-7x) + \cos(11x)\,\mathrm{d}x \\
&= \frac{1}{2}\left(\frac{\sin(-7x)}{-7} + \frac{\sin(11x)}{11}\right) + C \\
&= \boxed{-\frac{\sin(-7x)}{14} + \frac{\sin(11x)}{22} + C}
\end{aligned}$$

(f) $\int \frac{2x}{\sqrt{x^2-9}}\,\mathrm{d}x$

$$u=x^2-9,\quad \mathrm{d}u=2x\,\mathrm{d}x$$

$$\begin{aligned}\int \frac{2x}{\sqrt{x^2-9}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= 2\sqrt{u}+C \\ &= \boxed{2\sqrt{x^2-9}+C}\end{aligned}$$

(g) $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,\mathrm{d}x$

$$\overset{x=2\sin\theta,\quad \mathrm{d}x=2\cos\theta\,\mathrm{d}\theta}{}$$

$$\begin{aligned}\int \frac{x^3}{\sqrt{4-x^2}}\,\mathrm{d}x &= \int \frac{(2\sin\theta)^3}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,\mathrm{d}\theta \\ &= 8\int \sin^3\theta\,\mathrm{d}\theta \\ &= 8\int (1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta\end{aligned}$$

$$\overset{u=\cos\theta,\quad \mathrm{d}u=-\sin\theta\,\mathrm{d}\theta}{}$$

$$\begin{aligned}8\int (1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta &= 8\int u^2-1\,\mathrm{d}u \\ &= \frac{8u^3}{3}-8u+C \\ &= \frac{8\cos^3\theta}{3}-8\cos\theta+C \\ &= \frac{8\cos^3\arcsin\frac{x}{2}}{3}-8\cos\arcsin\frac{x}{2}+C \\ &= \frac{8\left(\frac{\sqrt{4-x^2}}{2}\right)^3}{3}-\frac{8\sqrt{4-x^2}}{2}+C \\ &= \boxed{\frac{1}{3}\sqrt{4-x^2}^3-4\sqrt{4-x^2}+C}\end{aligned}$$

(h) $\displaystyle\int \frac{-2x^3-6x^2-2x+10}{x^3+4x^2+5x}\,\mathrm{d}x$

$$\frac{-2x^3-6x^2-2x+10}{x^3+4x^2+5x} = -2+\frac{2x^2+8x+10}{x^3+4x^2+5x} = -2+\frac{2(x^2+4x+5)}{x(x^2+4x+5)} = -2+\frac{2}{x}$$

$$\int \frac{-2x^3-6x^2-2x+10}{x^3+4x^2+5x}\,\mathrm{d}x = \int -2+\frac{2}{x}\,\mathrm{d}x = \boxed{-2x+2\ln|x|+C}$$

5. Evaluate the following definite integrals

(a) $\displaystyle\int_1^2 xe^x\,\mathrm{d}x$

$$\overset{u=x,\quad \mathrm{d}u=\mathrm{d}x,\quad \mathrm{d}v=e^x\,\mathrm{d}x,\quad v=e^x}{}$$

$$\begin{aligned}\int_1^2 xe^x \, \mathrm{d}x &= xe^x\Big|_1^2 - \int_1^2 e^x \, \mathrm{d}x \\ &= xe^x - e^x\Big|_1^2 \\ &= 2e^2 - e^2 - (e - e) \\ &= \boxed{e^2}\end{aligned}$$

(b) $\int_0^{\frac{1}{2}} \sqrt{1 - x^2} \, \mathrm{d}x$

$$\begin{aligned}\int_0^{\frac{1}{2}} \sqrt{1-x^2}\, \mathrm{d}x &\overset{x=\sin\theta,\ \ \mathrm{d}x=\cos\theta \mathrm{d}\theta}{=} \int_{\arcsin 0}^{\arcsin \frac{1}{2}} \sqrt{1-\sin^2\theta} \cos\theta \, \mathrm{d}\theta \\ &= \int_0^{\frac{\pi}{6}} \cos^2\theta \, \mathrm{d}\theta \\ &= \frac{1}{2}\int_0^{\frac{\pi}{6}} 1 + \cos 2\theta \, \mathrm{d}\theta \\ &= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\Bigg|_0^{\frac{\pi}{6}} \\ &= \frac{1}{2}\left(\frac{\pi}{6} + \frac{\sin\frac{\pi}{3}}{2}\right) - \frac{1}{2}\left(0 + \frac{\sin 0}{2}\right)\Bigg|_0^{\frac{\pi}{6}} \\ &= \boxed{\frac{\pi}{12} + \frac{\sqrt{3}}{8}}\end{aligned}$$

6. Let

$$F(x) = \int_{e^x}^0 \arctan(t + t^2)\, \mathrm{d}t.$$

Find $F'(x)$.

$$F(x) = -\int_0^{e^x} \arctan(t + t^2)\, \mathrm{d}t$$

so

$$F'(x) = \boxed{-e^x \arctan(e^x + e^{2x})}$$

7. *(10 points)* Consider the bounded region between the curves $y^2 = x$ and $x^2 = y$.

(a) Find the area of the region.

If $y^2 = x$ and $x^2 = y$ then $(x^2)^2 = x$ so $x^4 - x = 0$. Thus $x = 0$ or $x = 1$.

$$\begin{aligned}
\text{Area} &= \int_0^1 \sqrt{x} - x^2 \, \mathrm{d}x \\
&= \frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\bigg|_0^1 \\
&= \frac{2 \cdot 1^{\frac{3}{2}}}{3} - \frac{1^3}{3} - \frac{2 \cdot 0^{\frac{3}{2}}}{3} + \frac{0^3}{3} \\
&= \boxed{\frac{1}{3}}
\end{aligned}$$

(b) Using washers give an integral representing the volume when this region is rotated about the line $y = 4$.

$$\begin{aligned}
\text{Volume} &= \int_0^1 \pi[R(x)]^2 - \pi[r(x)]^2 \, \mathrm{d}x \\
&= \boxed{\int_0^1 \pi(4 - x^2)^2 - \pi(4 - \sqrt{x})^2 \, \mathrm{d}x}
\end{aligned}$$

(c) Using cylindrical shells give an integral representing the volume when this region is rotated about the line $x = -2$.

$$\begin{aligned}
\text{Volume} &= \int_0^1 2\pi(x - (-2))(\sqrt{x} - x^2) \, \mathrm{d}x \\
&= \boxed{\int_0^1 2\pi(x + 2)(\sqrt{x} - x^2) \, \mathrm{d}x}
\end{aligned}$$

8. How much work is needed to retract a dangling $500\,\mathrm{m}$ chain with a density of $200\,\mathrm{kg/m}$?

Force of gravity acting on a segment of length δh is

$$\begin{aligned}
\text{Force} &= ma \\
&= 200\delta h(9.8)
\end{aligned}$$

$$\text{Work} \approx \sum_{i=1}^{n} h \cdot 200 \cdot \Delta h \cdot (9.8)$$

$$\begin{aligned}
\text{Work} &= \int_0^{500} 1960h \, \mathrm{d}h \\
&= h^2 1960\Big|_0^{500} \\
&= \boxed{1960 \cdot 500^2}
\end{aligned}$$

9. Find the maximum volume of a cylinder contained in a sphere of radius 1.

Let r be the radius of the cylinder and h be the height of the cylinder. If the cylinder fits inside the sphere then

$$r^2 + \left(\frac{h}{2}\right)^2 = 1$$

so

$$r^2 = 1 - \left(\frac{h}{2}\right)^2$$

$$V = \pi r^2 h = \pi\left(1 - \left(\frac{h}{2}\right)^2\right) h = \pi h - \frac{\pi}{4}h^3 \quad h \in [0,2]$$

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \pi - \frac{3\pi}{4}h^2$$

$$0 = \pi - \frac{3\pi}{4}h^2$$

$$h = \pm\frac{2}{\sqrt{3}}$$

V is a continuous function of h so by the Extreme Value Theorem it must have a maximum at its endpoints or critical points.

$$V(0) = \pi \cdot 0 - \frac{\pi}{4} \cdot 0^3 = 0$$

$$V(2) = \pi \cdot 2 - \frac{\pi}{4} \cdot 2^3 = 0$$

$$V\left(\frac{2}{\sqrt{3}}\right) = \pi \cdot \frac{2}{\sqrt{3}} - \frac{\pi}{4} \cdot \left(\frac{2}{\sqrt{3}}\right)^3 = \boxed{\frac{4\pi}{3\sqrt{3}}}$$

10. Water is draining from a spherical tank with radius r (in meters)

(a) If the water level is at a height of h meters above the center of the tank express the volume of water in the tank **as an integral** involving h.

The circular cross section of the tank at height y has radius $\sqrt{r^2 - y^2}$ so the volume of water is:

$$V = \int_{-r}^{h} \pi\sqrt{r^2 - y^2}^2 \,\mathrm{d}y = \boxed{\int_{-r}^{h} \pi(r^2 - y^2)\,\mathrm{d}y}$$

(b) If water is draining from the tank at a rate of $2\,\mathrm{m}^3/\mathrm{s}$ then at what rate is the water level changing when the water is $3\,\mathrm{m}$ above the center of the tank?

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-r}^{h} \pi(r^2 - y^2)\,\mathrm{d}y$$
$$= \pi(r^2 - h^2)\frac{\mathrm{d}h}{\mathrm{d}t}$$

so

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \pi(r^2 - h^2)\frac{\mathrm{d}h}{\mathrm{d}t}$$
$$-2 = \pi(r^2 - 3^2)\frac{\mathrm{d}h}{\mathrm{d}t}$$
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \boxed{-\frac{2}{\pi(r^2 - 3^2)}}$$

11. Find the length of the curve $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1$ using the arc length formula.

$$y' = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$
$$L = \int_0^1 \sqrt{1 + y'^2}\,\mathrm{d}x$$
$$= \int_0^1 \sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,\mathrm{d}x$$
$$= \int_0^1 \sqrt{1 + \frac{x^2}{1-x^2}}\,\mathrm{d}x$$
$$= \int_0^1 \sqrt{\frac{1 - x^2 + x^2}{1-x^2}}\,\mathrm{d}x$$
$$= \int_0^1 \sqrt{\frac{1}{1-x^2}}\,\mathrm{d}x$$
$$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$$= \arcsin x\Big|_0^1$$
$$= \arcsin 1 - \arcsin 0$$
$$= \boxed{\frac{\pi}{2}}$$

12. Consider the curve $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = \frac{1}{2}$

(a) Give an integral but **do not evaluate** for the surface area when the curve is rotated about the line $x = 2$

$y = \sqrt{1-x^2}$ so $x^2 + y^2 = 1$. Thus $x = \sqrt{1-y^2}$
If $x = 0$ then $y = \sqrt{1-0^2} = 1$
If $x = \frac{1}{2}$ then $y = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$

$$x' = \frac{-2y}{2\sqrt{1-y^2}} = \frac{-y}{\sqrt{1-y^2}}$$

$$SA = \boxed{\int_{\frac{\sqrt{3}}{2}}^{1} 2\pi(2 - \sqrt{1-y^2})\sqrt{1 + \left(\frac{-y}{\sqrt{1-y^2}}\right)^2}\, \mathrm{d}y}$$

(b) Give an integral but **do not evaluate** for the surface area when the curve is rotated about the line $y = 2$

$$y' = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$SA = \boxed{\int_{0}^{\frac{1}{2}} 2\pi(2 - \sqrt{1-x^2})\sqrt{1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\, \mathrm{d}x}$$