Practice Midterm 1 Solutions

1. **True or False** (you do not need to justify your answers)

   (a) **F**____ If f and g are differentiable at a then $f \circ g$ is differentiable at a.

   (b) **F**____ $\lim_{x \to \infty} \arctan x = 1$.

   (c) **T**____ If $f'(c) = 0$ and $g'(c) = 0$ then $(f \cdot g)'(c) = 0$.

   (d) **T**____ If $f(0) = 10$, $f(7) = 4$, and f is continuous on the closed interval $[-1, 8]$ the there must be some number $c \in (-1, 8)$ such that $f(c) = 9$.

   (e) **F**____ If f' has a vertical asymptote at c then f must have a vertical asymptote

2. Let

$$f(x) = \frac{1}{2x}.$$

   (a) What is $f'(x)$?

$$f'(x) = -\frac{1}{2x^2}$$

   (b) Using **only the definition of the derivative** compute $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\frac{1}{2(x+h)} - \frac{1}{2x}}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{x-(x+h)}{2x(x+h)}\right)}{h} \\ &= \lim_{h \to 0} \frac{\left(\frac{-h}{2x(x+h)}\right)}{h} \\ &= \lim_{h \to 0} \frac{-1}{2x(x+h)} \\ &= -\frac{1}{2x^2} \end{aligned}$$

3. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following functions

   (a) $y = (10 - e)\sqrt[5]{x^7} - \dfrac{4}{\sqrt[3]{\pi + 4}} + e^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{7}{5}(10 - e)x^{\frac{2}{5}}$$

   (b) $y = (2x)^{3x}$

$$\begin{aligned} y &= e^{3x \ln(2x)} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= e^{3x \ln(2x)} \left(3\ln(2x) + 3x \cdot \frac{2}{2x}\right) \\ &= (2x)^{3x} \left(3\ln(2x) + 3\right) \end{aligned}$$

   (c) $y = 20 \arcsin\left(\dfrac{2 - 4x}{x^3 + 1}\right)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{20}{\sqrt{1 - \left(\frac{2-4x}{x^3+1}\right)^2}} \cdot \frac{-4(x^3 + 1) - (2 - 4x)3x^2}{(x^3 + 1)^2}$$

4. Show that

$$\lim_{x\to -2} -\tfrac{1}{4}x^3 + 1 = 3$$

using **only the limit theorems**.

$$\begin{aligned}
\lim_{x\to -2}\left(-\tfrac{1}{4}x^3 + 1\right) &= \lim_{x\to -2} -\tfrac{1}{4}x^3 + \lim_{x\to -2} 1 \\
&= -\tfrac{1}{4}\left(\lim_{x\to -2} x\right)^3 + 1 \\
&= -\tfrac{1}{4}(-2)^3 + 1 \\
&= 3
\end{aligned}$$

5. Evaluate the following limits using any technique you like.

(a) $\displaystyle\lim_{x\to 0} \frac{3x^2+7x}{2x+5}$

$$\lim_{x\to 0} \frac{3x^2+7x}{2x+5} = \frac{3\cdot 0^2 + 7\cdot 0}{2\cdot 0 + 5} = 0$$

(b) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x^2+\sqrt[3]{2x^6+1}}}{x+5}$

$$\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{x^2+\sqrt[3]{2x^6+1}}}{x+5} &= \lim_{x\to\infty} \frac{\sqrt{x^2+\sqrt[3]{2x^6+1}}}{x+5}\cdot\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}\sqrt[3]{2x^6+1}}}{\frac{x}{x}+\frac{5}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{1+\sqrt[3]{\frac{2x^6}{x^6}+\frac{1}{x^6}}}}{1+\frac{5}{x}} \\
&= \lim_{x\to\infty} \frac{\sqrt{1+\sqrt[3]{2+\frac{1}{x^6}}}}{1+\frac{5}{x}} \\
&= \frac{\sqrt{1+\sqrt[3]{2+0}}}{1+0} \\
&= \sqrt{1+\sqrt[3]{2}}
\end{aligned}$$

6. Consider the curve $y^3 = x^4y - 3x + 9$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y

$$\begin{aligned}
y^3 &= x^4y - 3x + 9 \\
3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3y + x^4\frac{\mathrm{d}y}{\mathrm{d}x} - 3 \\
3y^2\frac{\mathrm{d}y}{\mathrm{d}x} - x^4\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3y - 3 \\
\frac{\mathrm{d}y}{\mathrm{d}x}(3y^2 - x^4) &= 4x^3y - 3 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4x^3y-3}{3x^2-x^4}
\end{aligned}$$

(b) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ solely in terms of x and y

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x^3y-3}{3x^2-x^4}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\left(12x^2y+4x^3\frac{\mathrm{d}y}{\mathrm{d}x}\right)(3x^2-x^4)-(4x^3y-3)(6x-4x^3)}{(3x^2-x^4)^2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\left(12x^2y+4x^3\cdot\frac{4x^3y-3}{3x^2-x^4}\right)(3x^2-x^4)-(4x^3y-3)(6x-4x^3)}{(3x^2-x^4)^2}$$

(c) Find the equation for the tangent line to the curve through the point $(1, 2)$

$$\text{slope} = \frac{4\cdot 1^3\cdot 2-3}{3\cdot 1^2-1^4} = \frac{5}{2}$$

$$y-2 = \frac{5}{2}(x-1)$$

7. A particle's position at time t is given by the equation

$$s(t) = \cos(p\pi t).$$

(a) Find the average velocity of the particle on the time interval from $t = 0$ to $t = \frac{2}{p}$

$$\begin{aligned}
\text{average velocity} &= \frac{\text{change in position}}{\text{change in time}} \\
&= \frac{s\left(\frac{2}{p}\right)-s(0)}{\frac{2}{p}-0} \\
&= \frac{\cos\left(p\cdot\pi\cdot\frac{2}{p}\right)-\cos(p\cdot\pi\cdot 0)}{\frac{2}{p}} \\
&= \frac{\cos(2\pi)-\cos(0)}{\frac{2}{p}} \\
&= \frac{1-1}{\frac{2}{p}} \\
&= 0
\end{aligned}$$

(b) Give an equation for the acceleration of the particle at time t.

$$\begin{aligned}
a(t) &= \frac{\mathrm{d}^2}{\mathrm{d}t^2}\cos(p\pi t) \\
&= \frac{\mathrm{d}}{\mathrm{d}t} - p\pi\sin(p\pi t) \\
&= -p^2\pi^2\cos(p\pi t)
\end{aligned}$$

8. Simplify the following expressions

(a) $\sec(\tan^{-1} x)$

$$\begin{aligned}
\sec(\tan^{-1} x) &= \sqrt{1+[\tan(\tan^{-1} x)]^2} \\
&= \sqrt{1+x^2}
\end{aligned}$$

(b) $\sin(\cos^{-1} x)$

$$\begin{aligned}\sin(\cos^{-1} x) &= \sqrt{1 - [\cos(\cos^{-1} x)]^2} \\ &= \sqrt{1 - x^2}\end{aligned}$$