Math 161.01 November 17, 2010

Practice Midterm 2

- Calculators are allowed as long as they have no symbolic integration capbility (TI-84 and comparable are ok)
- Remember to **CIRCLE YOUR FINAL ANSWER.**

1. **True or False** (you do not need to justify your answers)

   (a) **F**____ If $f'(c) = 0$ then c must be a local minimum or a local maximum for f.

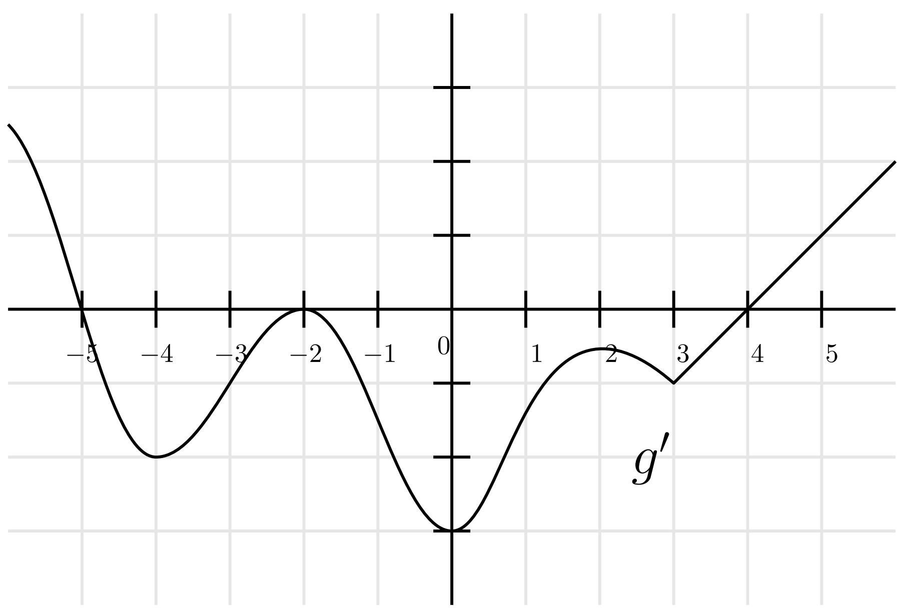
   (b) **T**____ If f is continuous on the closed interval $[0, 7]$ then f has a global maximum and a global mininum on $[0, 7]$.

   (c) **T**____ 0 is both a critical point and an inflection point of $f(x) = x^3$.

   (d) **T**____ Suppose f is continuous on $\mathbb{R}$. If f is decreasing on the interval $[2, 5]$ and decreasing on the interval $[5, 10]$ then it is decreasing on the interval $[2, 10]$.

   (e) **F**____ If f' has a vertical asymptote at c then f must have a vertical asymptote

2. The **derivative** of a function g is pictured below.



   (a) List the critical points of g.

   $-5, -2$ and 4

   (b) List the inflection points of g.

   $-4, -2, 0, 2$ and 3

   (c) List the interval(s) on which g is **decreasing**. Be sure that your intervals are as large as possible.

   $[-5, 4]$

(d) List the interval(s) on which g is **concave up**.

$\boxed{[-4,-2],\ [0,2] \text{ and } [3,\infty)}$

3. Evaluate the following limits using any technique you like.

(a) $\lim\limits_{x\to 0} \dfrac{3x^2}{\sin x}$

$$\begin{aligned}\lim_{x\to 0}\frac{3x^2}{\sin x} &= \lim_{x\to 0}\frac{6x}{\cos x}\\ &= \lim_{x\to 0}\frac{0}{1}\\ &= \boxed{0}\end{aligned}$$

(b) $\lim\limits_{x\to\infty} \sqrt[x]{x}$

$$\begin{aligned}\lim_{x\to\infty}\sqrt[x]{x} &= \lim_{x\to\infty} x^{\frac{1}{x}}\\ &= \lim_{x\to\infty} e^{\frac{\ln x}{x}}\\ &= e^{\lim_{x\to\infty}\frac{\ln x}{x}}\\ &= e^{\lim_{x\to\infty}\frac{x^{-1}}{1}}\\ &= e^{\frac{0}{1}}\\ &= \boxed{1}\end{aligned}$$

(c) $\lim\limits_{x\to\infty}\left(1+\dfrac{2}{x}\right)^x$

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x &= \lim_{x\to\infty} e^{x\ln\left(1+\frac{2}{x}\right)}\\ &= e^{\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right)}\\ &= e^{\lim_{x\to\infty}\frac{\ln\left(1+\frac{2}{x}\right)}{x^{-1}}}\\ &= e^{\lim_{x\to\infty}\frac{\left(1+\frac{2}{x}\right)^{-1}(-2)x^{-2}}{(-1)x^{-2}}}\\ &= e^{\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{-1}(2)}\\ &= e^{(1^{-1}\cdot 2)}\\ &= \boxed{e^2}\end{aligned}$$

4. Evaluate the following indefinite integrals.

(a) $\displaystyle\int \sec^2(6x-2)\,\mathrm{d}x$

Let $u = 6x-2$ so $\mathrm{d}u = 6\mathrm{d}x$.

$$\begin{aligned}\int \sec^2(6x-2)\,\mathrm{d}x &= \int \sec^2(u)\,\frac{1}{6}\,\mathrm{d}u\\ &= \frac{1}{6}\tan(u)+C\\ &= \boxed{\tfrac{1}{6}\tan(6x-2)+C}\end{aligned}$$

CHECK

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{6}\tan(6x-2)+C\right) = \left(\frac{1}{6}\sec^2(6x-2)\right)\cdot 6+0$$
$$= \sec^2(6x-2) \quad ✓$$

(b) $\displaystyle\int 2^x - x^2 - \frac{2}{x}\,\mathrm{d}x$
Let $u = 6x-2$ so $\mathrm{d}u = 6\mathrm{d}x$.

$$\int 2^x - x^2 - \frac{2}{x}\,\mathrm{d}x = \int e^{x\ln 2} - x^2 - 2x^{-1}\,\mathrm{d}u$$
$$= \frac{e^{x\ln 2}}{\ln 2} - \frac{x^3}{3} - 2\ln x + C$$
$$= \boxed{\frac{2^x}{\ln 2} - \frac{x^3}{3} - 2\ln x + C}$$

CHECK

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2^x}{\ln 2} - \frac{x^3}{3} - 2\ln x + C\right) = \ln 2\cdot\frac{2^x}{\ln 2} - 3\cdot\frac{x^2}{3} - \frac{2}{x} + 0$$
$$= 2^x - x^2 - \frac{2}{x} \quad ✓$$

5. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^1 \frac{5}{1+x^2}\,\mathrm{d}x$

$$\int_0^1 \frac{5}{1+x^2}\,\mathrm{d}x = 5\arctan x\Big|_0^1$$
$$= 5\arctan 1 - 5\arctan 0$$
$$= \boxed{\tfrac{5\pi}{4}}$$

(b) $\displaystyle\int_1^2 \sqrt{x+5}\,\mathrm{d}x$
Let $u = x+5$ so $\mathrm{d}u = \mathrm{d}x$.

$$\int_1^2 \sqrt{x+5}\,\mathrm{d}x = \int_{1+5}^{2+5} \sqrt{u}\,\mathrm{d}u$$
$$= \frac{2u^{\frac{3}{2}}}{3}\Bigg|_6^7$$
$$= \boxed{\frac{2\cdot 7^{\frac{3}{2}}}{3} - \frac{2\cdot 6^{\frac{3}{2}}}{3}}$$

(c) Let

$$F(x) = \int_{x^2}^{\ln x} t^2\cot(t^3+8)\,\mathrm{d}t.$$

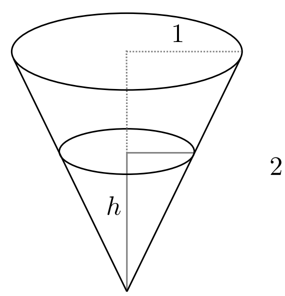
Find $F'(x)$.

$$
\begin{aligned}
F(x) &= \int_{x^2}^{\ln x} t^2 \cot(t^3+8)\,\mathrm{d}t \\
&= \int_{x^2}^{1} t^2 \cot(t^3+8)\,\mathrm{d}t + \int_{1}^{\ln x} t^2 \cot(t^3+8)\,\mathrm{d}t \\
&= -\int_{1}^{x^2} t^2 \cot(t^3+8)\,\mathrm{d}t + \int_{1}^{\ln x} t^2 \cot(t^3+8)\,\mathrm{d}t
\end{aligned}
$$

Hence,

$$
\boxed{F'(x) = -2x(x^2)^2 \cot((x^2)^3+8) + \frac{1}{x}(\ln x)^2 \cot((\ln x)^3+8)}
$$

6. Water is pouring into a cone with a height of 2 meters and a radius of 1 meter at a rate of $3\ \mathrm{m}^3/\mathrm{s}$.



(a) Give an equation relating the height h of the water to the volume V of water in the cone. *(Hint: The volume of a cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.)*
By similar triangles,

$$\frac{h}{r} = \frac{2}{1}$$

so

$$r = \frac{h}{2}$$

Therefore,

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h$$

$$\boxed{V = \frac{\pi}{12}h^3}$$

(b) When the height of the water is 1 meter how fast is the water level rising.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(V = \frac{\pi}{12}h^3\right)$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\pi}{12}\cdot 3h^2 \frac{\mathrm{d}h}{\mathrm{d}t}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t}\frac{4}{\pi h^2}$$

$\frac{dV}{dt} = 3$ and $h = 1$ so

$$\begin{aligned}\frac{dh}{dt} &= 3 \cdot \frac{4}{\pi \cdot 1^2} \\ &= \boxed{\tfrac{12}{\pi}\mathrm{m/s}}\end{aligned}$$

7. The height h and radius r of a circular cylinder are both greater than or equal to 0 and have a sum of 1 meter.

   (a) Express the volume V of the cylinder as a function of its radius r. Include the **domain** for the radius. *(Hint: The volume of a cylinder with height h and radius r is $\pi r^2 h$.)*

   $$\begin{aligned}h + r &= 1 \\ h &= 1 - r \\ V &= \pi r^2 h\end{aligned}$$

   so

   $$V = \pi r^2(1 - r)$$

   For the domain, $h = 1 - r \geq 0$ so $1 \geq r$. Also $r \geq 0$ so $0 \leq r \leq 1$.

   $$\boxed{V = \pi r^2 - \pi r^3 \quad r \in [0, 1]}$$

   (b) Is there a radius in the domain above which gives the maximum volume for the cylinder? Why or why not? In particular do any theorems from class or the book apply to this situation?

   Volume is a continuous function of the radius on the closed interval $[0, 1]$ so by the Extreme Value Theorem, there is a radius in the domain $[0, 1]$ which gives the maximum volume for the cylinder.

   (c) Find the radius r of the cylinder with the maximum volume.

   $$\frac{dV}{dr} = 2\pi r - 3\pi r^2 = \pi r(2 - 3r)$$

   Critical points satisfy $0 = \pi r(2 - 3r)$. So the critical points are 0 and $\frac{2}{3}$.
   Endpoints are 0 and 1.

   $$\begin{aligned}V(0) &= \pi 0^2 - \pi 0^3 = 0 \\ V\left(\frac{2}{3}\right) &= \pi\left(\frac{2}{3}\right)^2 - \pi\left(\frac{2}{3}\right)^3 = \frac{4\pi}{27} \\ V(1) &= \pi 1^2 - \pi 1^3 = 0\end{aligned}$$

   The maximum volume occurs when the radius is $\boxed{\frac{2}{3}\mathrm{m}}$.

   (d) What is the maximum volume?

   The maximum volume is $\boxed{\frac{4\pi}{27}\mathrm{m}^3}$.