Practice Final Exam – Math 2153

1. Decide if the following statements are TRUE or FALSE and circle your answer. **You do NOT need to justify your answers.**

   (a) (1 point) If line integrals in the continuous vector field $\mathbf{F}(x,y)$ are path independent then $\mathbf{F}$ is a conservative vector field.

   Solution: **T** (Path independence of line integrals in $\mathbf{F}$ is equivalent to $\mathbf{F}$ being conservative.)

   (b) (1 point) If $\mathbf{F}$ is a conservative vector field then line integrals in the continuous vector field $\mathbf{F}(x,y)$ are path independent.

   Solution: **T** (Path independence of line integrals in $\mathbf{F}$ is equivalent to $\mathbf{F}$ being conservative.)

   (c) (1 point) If $\mathbf{F}(x,y)$ has continuous first partial derivatives on the connected, simply connected region R and $\mathbf{F}(x,y)$ is irrotational then $\mathbf{F}$ is conservative.

   Solution: **T** (On a connected, simply connected domain $\mathbf{F}$ is irrotational if and only if $\mathbf{F}$ is conservative.)

   (d) (1 point) If $\mathbf{F}(x,y)$ has continuous first partial derivatives on the connected, simply connected region R and $\mathbf{F}(x,y)$ is source-free then $\mathbf{F}$ is conservative.

   Solution: **F** (The vector field $\mathbf{F} = \langle -y, x\rangle$ is source-free since the 2-dimensional divergence of $\mathbf{F}$ is 0. However $\mathbf{F}$ is not irrotational since the 2-dimensional curl of $\mathbf{F}$ is 2.)

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

   (a) (2 points) Give an example of a scalar function $f(x,y)$ whose implicit domain is connected but not simply connected.

   Solution: $\boxed{f(x,y) = \frac{1}{x^2+y^2}}$

   (b) (2 points) Give an example of a non-constant conservative vector field $\mathbf{F}(x,y,z)$ with domain $\mathbf{R}^3$.

   Solution: Let $\varphi(x,y,z) = x^2y - z^3$. Then $\nabla\varphi = \langle 2xy, x^2, 3z^2\rangle$. $\boxed{\mathbf{F}(x,y,z) = \langle 2xy, x^2, 3z^2\rangle}$

   (c) (2 points) Give an example of parametrized path in $\mathbf{R}^2$ which is not a simple path.

   Solution: $\boxed{\mathbf{r}(t) = \langle \cos t, \sin t\rangle \text{ where } 0 \le t \le 3\pi}$

   (d) (2 points) Give an example of a non-constant source-free vector field $\mathbf{F}(x,y,z)$ with domain $\mathbf{R}^3$.

   Solution: Let $\mathbf{G}(x,y,z) = \langle z, yz, xyz\rangle$. Then $\nabla \times \mathbf{G} = \langle xz - z, 1 - yz, 0\rangle$.
   Let $\boxed{\mathbf{F}(x,y,z) = \langle xz - z, 1 - yz, 0\rangle}$

3. Compute the following line integrals using any technique you like:

(a) (5 points) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y) = \langle xy, x - y \rangle$ and C is the straight line segment from the point $(0, 0)$ to the point $(2, 1)$.

Solution: First we parametrize C as $\mathbf{r}(t) = (1-t)\langle 0, 0\rangle + t\langle 2, 1\rangle = \langle 2t, t\rangle$ where $0 \le t \le 1$.

$$\mathbf{r}'(t) = \langle 2, 1 \rangle$$
$$\mathbf{F}(\mathbf{r}(t)) = \langle 2t^2, t \rangle$$

Thus,

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\
&= \int_0^1 \langle 2t^2, t\rangle \cdot \langle 2, 1\rangle\, dt \\
&= \int_0^1 4t^2 + t\, dt \\
&= \frac{4t^3}{3} + \frac{t^2}{2}\Big|_{t=0}^{1} \\
&= \boxed{\tfrac{4}{3} + \tfrac{1}{2}}
\end{aligned}$$

(b) (2 points) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F}(x, y, z) = \langle 6xyz, 3x^2 z, 3x^2 y\rangle$ and C is the path with parametrization

$$\mathbf{r}(t) = \langle t, \sin t, t \sin t\rangle \qquad 0 \le t \le \pi$$

Solution:

$$\begin{aligned}
\operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xyz & 3x^2 z & 3x^2 y \end{vmatrix} \\
&= (3x^2 - 3x^2)\,\mathbf{i} - (6xy - 6xy)\,\mathbf{j} + (6xz - 6xz)\,\mathbf{k} \\
&= \mathbf{0}
\end{aligned}$$

Domain of $\mathbf{F}$ is $\mathbf{R}^3$ which is connected and simply connected. hence $\mathbf{F}$ is conservative. Let $\varphi(x, y, z)$ be a potential function for $\mathbf{F}$. Then

$$\varphi_x = 6xyz, \qquad \varphi_y = 3x^2 z, \qquad \varphi_z = 3x^2 y.$$

Hence

$$\begin{aligned}
\varphi &= \int 6xyz\, dx = 3x^2 yz + A(y, z) \\
3x^2 z &= 3x^2 z + A_y(y, z)
\end{aligned}$$

$$0 = A_y(y,z)$$

Therefore

$$\begin{aligned} A(y,z) &= \int 0\,dy = B(z) \\ \varphi &= 3x^2yz + A(y,z) = 3x^2yz + B(z) \\ 3x^2y &= 3x^2y + B'(z) \\ 0 &= B'(z) \end{aligned}$$

Therefore

$$\begin{aligned} B(z) &= \int 0\,dz = 0 \\ \varphi &= 3x^2yz + B(z) = 3x^2yz \end{aligned}$$

Check:

$$\nabla\varphi = \langle 6xyz, 3x^2z, 3x^2y\rangle = \mathbf{F}$$

Thus

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= \varphi(\mathbf{r}(\pi)) - \varphi(\mathbf{r}(0)) \\ &= \varphi(\pi, \sin\pi, \pi\sin\pi) - \varphi(0, \sin 0, 0\sin 0) \\ &= \varphi(\pi,0,0) - \varphi(0,0,0) \\ &= 3\pi^2\cdot 0\cdot 0 - 3\cdot 0^2\cdot 0\cdot 0 \\ &= \boxed{0} \end{aligned}$$

(c) (2 points) Evaluate

$$\oint_C \mathbf{F}\cdot d\mathbf{r}$$

where $\mathbf{F}(x,y) = \langle xy^2, x^2 - y\rangle$ and C is the closed square path with corners $(0,0)$, $(0,2)$, $(2,2)$ and $(2,0)$ oriented *clockwise*.

Solution: Let $R = \{(x,y)\,|\,0 \le x \le 2 \text{ and } 0 \le y \le 2\}$. Then by the circulation version of Green's Theorem

$$\begin{aligned} \oint_C \mathbf{F}\cdot d\mathbf{r} &= -\iint_R \text{2D-curl } \mathbf{F}\,dA \\ &= -\int_0^2\int_0^2 \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\,dy\,dx \\ &= -\int_0^2\int_0^2 2x - 2xy\,dy\,dx \\ &= -\int_0^2 2xy - xy^2\Big|_{y=0}^2\,dx \end{aligned}$$

$$
\begin{aligned}
&= -\int_0^2 4x - 4x\,dx \\
&= -\int_0^2 0\,dx \\
&= \boxed{0}
\end{aligned}
$$

4. Let

$$\mathbf{F}(x, y, z) = \langle x^2y, xyz, z^2 \rangle$$

(a) (5 points) Compute

$$\operatorname{curl}\mathbf{F}$$

Solution:

$$
\begin{aligned}
\operatorname{curl}\mathbf{F} &= \nabla \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xyz & z^2 \end{vmatrix} \\
&= \boxed{-xy\,\mathbf{i} - 0\,\mathbf{j} + (yz - x^2)\,\mathbf{k}}
\end{aligned}
$$

(b) (5 points) Compute

$$\operatorname{div}\mathbf{F}$$

Solution:

$$
\begin{aligned}
\operatorname{div}\mathbf{F} &= \nabla \cdot \mathbf{F} \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x^2y, xyz, z^2 \rangle \\
&= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(z^2) \\
&= \boxed{2xy + xz + 2z}
\end{aligned}
$$

(c) (5 points) Compute

$$\operatorname{div}(\operatorname{curl}\mathbf{F})$$

Solution:

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \boxed{0}$$

for all vector fields $\mathbf{F}$.

5. Compute the following surface integrals using any technique you like:

(a) (5 points) Evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

where $\mathbf{F}(x,y,z) = \langle z, z, z \rangle$ and S is the upper half of the sphere or radius 2 with center $(0,0,0)$ oriented *inwards*.

Solution: First we parametrize S as

$$\mathbf{r}(u,v) = \langle 2\sin u\cos v, 2\sin u\sin v, 2\cos u \rangle$$

with domain $R = \{(u,v)\,|\,0 \le u \le \frac{\pi}{2} \text{ and } 0 \le v \le 2\pi\}$.

$$\begin{aligned}
\mathbf{t}_u &= \frac{\partial \mathbf{r}}{\partial u} = \langle 2\cos u\cos v, 2\cos u\sin v, -2\sin u \rangle \\
\mathbf{t}_v &= \frac{\partial \mathbf{r}}{\partial v} = \langle -2\sin u\sin v, 2\sin u\cos v, 0 \rangle \\
\mathbf{t}_u \times \mathbf{t}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2\cos u\cos v & 2\cos u\sin v & -2\sin u \\ -2\sin u\sin v & 2\sin u\cos v & 0 \end{vmatrix} \\
\mathbf{t}_u \times \mathbf{t}_v &= 4\sin^2 u\cos v\,\mathbf{i} + 4\sin^2 u\sin v\,\mathbf{j} + (4\sin u\cos u\cos^2 v + 4\sin u\cos u\sin^2 v)\,\mathbf{k} \\
\mathbf{t}_u \times \mathbf{t}_v &= 4\sin^2 u\cos v\,\mathbf{i} + 4\sin^2 u\sin v\,\mathbf{j} + 4\sin u\cos u\,\mathbf{k}
\end{aligned}$$

To get the inwards orientation for S we must take the negative of $\mathbf{t}_u \times \mathbf{t}_v$.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \mathbf{F} \cdot (-\mathbf{t}_u \times \mathbf{t}_v)\, dA \\
&= -\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \langle 2\cos u, 2\cos u, 2\cos u \rangle \cdot \langle 4\sin^2 u\cos v,\ 4\sin^2 u\sin v,\ 4\sin u\cos u \rangle\, dv\, du \\
&= -\int_0^{\frac{\pi}{2}} \int_0^{2\pi} 8\sin^2 u\cos u\cos v + 8\sin^2 u\cos u\sin v + 8\sin u\cos^2 u\, dv\, du \\
&= -\int_0^{\frac{\pi}{2}} 8\sin^2 u\cos u\sin v - 8\sin^2 u\cos u\cos v + 8v\sin u\cos^2 u\Big|_{v=0}^{2\pi}\, du \\
&= -\int_0^{\frac{\pi}{2}} 16\pi\sin u\cos^2 u\, du \\
&\quad \text{substitute: } w = \cos u, dw = -\sin u\, du \\
&= \int_{\cos 0}^{\cos\frac{\pi}{2}} 16\pi w^2\, dw \\
&= \int_1^0 16\pi w^2\, dw \\
&= \frac{16\pi w^3}{3}\Big|_{w=1}^{0} \\
&= \boxed{-\tfrac{16\pi}{3}}
\end{aligned}$$

(b) (2 points) Evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

where $\mathbf{F}(x, y, z) = \langle x + y + z, 0, 0\rangle$ and S is the upper half of the sphere or radius 2 with center $(0, 0, 0)$ oriented oriented *inwards*.

Solution: Let C be the curve

$$\mathbf{r}(t) = \langle -\cos t, \sin t, 0\rangle \qquad 0 \le t \le 2\pi.$$

By Stokes' Theorem

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \langle -\cos t + \sin t, 0, 0\rangle \cdot \langle \sin t, \cos t, 0\rangle\, dt \\
&= \int_0^{2\pi} -\cos t \sin t + \sin^2 t\, dt \\
&= \int_0^{2\pi} -\frac{1}{2}\sin 2t + \frac{1}{2}(1 - \cos 2t)\, dt \\
&= \frac{1}{4}\cos 2t + \frac{t}{2} - \frac{1}{4}\sin 2t\bigg|_{t=0}^{2\pi} \\
&= \boxed{\pi}
\end{aligned}$$

6. (5 points) Change the order of integration for the double integral $\int_0^{\pi}\int_0^{\sin x} xy - 2\, dy\, dx$.

Solution:

$$\int_0^{\pi}\int_0^{\sin x} xy - 2\, dy\, dx = \boxed{\int_0^1 \int_{\arcsin x}^{2\pi - \arcsin x} xy - 2\, dx\, dy}$$

7. (5 points) Evaluate the double integral

$$\iint_R \frac{2y}{1 + x^2}\, dA$$

where $R = \{(x, y) \,|\, 0 \le x \le 1 \text{ and } 0 \le y \le \sqrt{x}\}$.

Solution:

$$\begin{aligned}
\iint_R \frac{2y}{1 + x^2}\, dA &= \int_0^1 \int_0^{\sqrt{x}} \frac{2y}{1 + x^2}\, dy\, dx \\
&= \int_0^1 \frac{y^2}{1 + x^2}\bigg|_{y=0}^{\sqrt{x}}\, dx \\
&= \int_0^1 \frac{x}{1 + x^2}\, dx
\end{aligned}$$

substitute: $u = 1 + x^2, du = 2x\,dx$

$$= \int_{1+0^2}^{1+1^2} \frac{1}{2u}\,du$$

$$= \frac{1}{2}\ln(u)\Big|_{u=1}^{2}$$

$$= \boxed{\frac{\ln 2}{2}}$$

8. (5 points) Compute the maximum value for the function $f(x, y) = x - yx$ on the region

$$R = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$$

Solution: First we find the critical points of f.

$$\nabla f = \langle 1 - y, -x \rangle$$

Thus $0 = 1 - y$ and $0 = -x$. Hence $y = 1$ and $x = 0$. Thus the critical point of f is $(0, 1)$. This point is on the boundary of R so it should also appear again when we examine the boundary of R.

Now let $g(x, y) = x^2 + y^2 - 1$. We wish to maximize f subject to the constraint $g(x, y) = 0$. Applying the method of Lagrange multipliers we compute

$$\nabla g = \langle 2x, 2y \rangle$$

Setting $\nabla f = \lambda \nabla g$ gives the two equations

$$\begin{aligned} 1 - y &= 2\lambda x \\ -x &= 2\lambda y \end{aligned}$$

Substituting x from the second equation into the first equation we get

$$\begin{aligned} 1 - y &= -4\lambda^2 y \\ 4\lambda^2 y - y &= -1 \\ y &= \frac{1}{1 - 4\lambda^2} \\ x &= -2\lambda y = \frac{-2\lambda}{1 - 4\lambda^2} \end{aligned}$$

From the constraint equation we get $x^2 + y^2 = 1$ so

$$\begin{aligned} \left(\frac{-2\lambda}{1 - 4\lambda^2}\right)^2 + \left(\frac{1}{1 - 4\lambda^2}\right)^2 &= 1 \\ 4\lambda^2 + 1 &= (1 - 4\lambda^2)^2 \\ 4\lambda^2 + 1 &= 1 - 8\lambda^2 + 16\lambda^4 \\ 0 &= 16\lambda^4 - 12\lambda^2 \\ 0 &= 4\lambda^2(4\lambda^2 - 3) \end{aligned}$$

Thus $\lambda^2 = 0$ or $\lambda^2 = \frac{3}{4}$. Hence $\lambda \in \left\{0, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right\}$ which yields the points

$$(0,1) \qquad \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \qquad \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

$$\begin{aligned} f(0,1) &= 0 - 1(0) = 0 \\ f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) &= -\frac{\sqrt{3}}{2} - \left(-\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2}\right) = -\frac{3\sqrt{3}}{4} \\ f\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) &= \frac{\sqrt{3}}{2} - \left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \boxed{\frac{3\sqrt{3}}{4}} \end{aligned}$$

9. (5 points) Compute the curvature κ of the curve $y = \cos x$ at the point $(0, 1)$.

Solution: We can parametrize the curve $y = \cos x$ as

$$\mathbf{r}(t) = \langle t, \cos t\rangle$$

$$\begin{aligned} \mathbf{r}'(t) &= \langle 1, -\sin t\rangle \\ \mathbf{r}''(t) &= \langle 0, -\cos t\rangle \end{aligned}$$

Solving the second equation for λ we get $\lambda = \frac{-2y+x}{2y}$. Substituting this into the first equation we get

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -\sin t & 0 \\ 0 & -\cos t & 0 \end{vmatrix} \\ &= -\cos t\,\mathbf{k} \end{aligned}$$

$$\begin{aligned} \kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \\ &= \frac{|\cos t|}{|1^2 + \sin^2 t|^3} \\ \kappa(0) &= \boxed{1} \end{aligned}$$

10. (5 points) Find an equation for the tangent plane to the level surface of the scalar function

$$f(x, y, z) = x^2 y - z^2 x + 2$$

passing though the point $(1, 2, -1)$

Solution: A normal vector for the tangent plane is the gradient vector

$$\begin{aligned}\nabla f &= \left\langle 2xy - z^2, x^2, -2zx \right\rangle \\ \nabla f(1,2,-1) &= \left\langle 2(1)(2) - (-1)^2, (1)^2, -2(-1)(1) \right\rangle \\ \nabla f(1,2,-1) &= \langle 3, 1, 2 \rangle\end{aligned}$$

Thus the tangent plane satisfies the equation

$$\boxed{3(x-1) + (y-2) + 2(z+1) = 0}$$

11. Show that the following vector fields are not conservative on their implicit domains.

(a) (5 points) $\mathbf{F}(x,y,z) = \langle x^2y, 2y, z - x \rangle$

Solution:

$$\begin{aligned}\operatorname{curl} \mathbf{F} &= \nabla \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y & z - x \end{vmatrix} \\ &= 0\,\mathbf{i} - (-1)\,\mathbf{j} + (-x^2)\,\mathbf{k} \\ &= \mathbf{j} - x^2\,\mathbf{k}\end{aligned}$$

Thus $\mathbf{F}$ is not irrotational and cannot be conservative.

(b) (5 points) $\mathbf{F}(x,y) = \left\langle \dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2} \right\rangle$

Solution:

$$\begin{aligned}\text{2D-curl}\,\mathbf{F} &= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \\ &= \frac{2xy}{(x^2+y^2)^2} - \frac{-2xy}{(x^2+y^2)^2} \\ &= \frac{4xy}{(x^2+y^2)^2}\end{aligned}$$

Thus $\mathbf{F}$ is not irrotational and cannot be conservative.

12. Find potential functions for the following conservative vector fields

(a) (5 points) $\mathbf{F}(x,y) = \langle 1 - y\cos(xy), -x\cos(xy) \rangle$

Solution: A potential function $\varphi(x,y)$ for $\mathbf{F}$ must satisfy

$$\varphi_x = 1 - y\cos(xy), \qquad \varphi_y = -x\cos(xy).$$

Hence

$$\varphi = \int 1 - y\cos(xy)\,dx = x - \sin(xy) + A(y)$$

$$-x\cos(xy) = -x\cos(xy) + A'(y)$$
$$0 = A'(y)$$

Therefore

$$A(y) = \int 0\,dy = 0$$
$$\varphi = x - \sin(xy) + A(y) = \boxed{x - \sin(xy)}$$

Check:

$$\nabla\varphi = \langle 1 - y\cos(xy), -x\cos(xy)\rangle = \mathbf{F}$$

(b) (5 points) $\mathbf{F}(x, y, z) = \langle yz, xz + 3, xy + 2\rangle$

Solution: A potential function $\varphi(x, y)$ for $\mathbf{F}$ must satisfy

$$\varphi_x = yz, \quad \varphi_y = xz + 3, \quad \varphi_z = xy + 2.$$

Hence

$$\varphi = \int yz\,dx = xyz + A(y, z)$$
$$xz + 3 = xz + A_y(y, z)$$
$$3 = A_y(y, z)$$

Therefore

$$A(y, z) = \int 3\,dy = 3y + B(z)$$
$$\varphi = xyz + A(y, z) = xyz + 3y + B(z)$$
$$xy + 2 = xy + B'(z)$$
$$2 = B'(z)$$

Therefore

$$B(z) = \int 2\,dz = 2z$$
$$\varphi = xyz + 3y + B(z) = \boxed{xyz + 3y + 2z}$$

Check:

$$\nabla\varphi = \langle yz, xz + 3, xy + 2\rangle = \mathbf{F}$$

13. (5 points) Find the x-coordinate of the center of mass of a thin triangular plate with vertices $(0, 0)$, $(0, 1)$ and $(2, 0)$ with density function

$$f(x, y) = x + 1.$$

Solution: Let R be the triangular region

$$R = \{(x, y) \mid x \geq 0,\ y \geq 0 \text{ and } x + 2y \leq 2\}$$

The total mass is

$$\begin{aligned}
M &= \int_R x + 1\, dA \\
&= \int_0^2 \int_0^{1-\frac{x}{2}} x + 1\, dy\, dx \\
&= \int_0^2 y(x+1)\Big|_{y=0}^{1-\frac{x}{2}}\, dx \\
&= \int_0^2 \left(1 - \frac{x}{2}\right)(x+1)\, dx \\
&= \int_0^2 -\frac{x^2}{2} + \frac{x}{2} + 1\, dx \\
&= -\frac{x^3}{6} + \frac{x^2}{4} + x\Big|_{x=0}^{2} \\
&= -\frac{8}{6} + 1 + 2 \\
&= \frac{5}{3}
\end{aligned}$$

$$\begin{aligned}
M_x &= \int_R x(x+1)\, dA \\
&= \int_0^2 \int_0^{1-\frac{x}{2}} x^2 + x\, dy\, dx \\
&= \int_0^2 y(x^2+x)\Big|_{y=0}^{1-\frac{x}{2}}\, dx \\
&= \int_0^2 \left(1 - \frac{x}{2}\right)(x^2+x)\, dx \\
&= \int_0^2 -\frac{x^3}{2} + \frac{x^2}{2} + x\, dx \\
&= -\frac{x^4}{8} + \frac{x^3}{6} + \frac{x^2}{2}\Big|_{x=0}^{2} \\
&= -\frac{16}{8} + \frac{8}{6} + \frac{4}{2} \\
&= \frac{4}{3}
\end{aligned}$$

The x-coordinate of the centroid is

$$\frac{M_x}{M} = \boxed{\tfrac{4}{5}}$$