Practice Final Exam – Math 2153

1. Decide if the following statements are TRUE or FALSE and circle your answer. **You do NOT need to justify your answers.**
   (a) (1 point) If line integrals in the continuous vector field $\mathbf{F}(x, y)$ are path independent then $\mathbf{F}$ is a conservative vector field.
   (b) (1 point) If $\mathbf{F}$ is a conservative vector field then line integrals in the continuous vector field $\mathbf{F}(x, y)$ are path independent.
   (c) (1 point) If $\mathbf{F}(x, y)$ has continuous first partial derivatives on the connected, simply connected region R and $\mathbf{F}(x, y)$ is irrotational then $\mathbf{F}$ is conservative.
   (d) (1 point) If $\mathbf{F}(x, y)$ has continuous first partial derivatives on the connected, simply connected region R and $\mathbf{F}(x, y)$ is source-free then $\mathbf{F}$ is conservative.

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**
   (a) (2 points) Give an example of a scalar function $f(x, y)$ whose implicit domain is connected but not simply connected.
   (b) (2 points) Give an example of a non-constant conservative vector field $\mathbf{F}(x, y, z)$ with domain $\mathbf{R}^3$.
   (c) (2 points) Give an example of parametrized path in $\mathbf{R}^2$ which is not a simple path.
   (d) (2 points) Give an example of a non-constant source-free vector field $\mathbf{F}(x, y, z)$ with domain $\mathbf{R}^3$.

3. Compute the following line integrals using any technique you like:
   (a) (5 points) Evaluate
   $$\int_C \mathbf{F} \cdot d\mathbf{r}$$
   where $\mathbf{F}(x, y) = \langle xy, x - y \rangle$ and C is the straight line segment from the point $(0, 0)$ to the point $(2, 1)$.
   (b) (2 points) Evaluate
   $$\int_C \mathbf{F} \cdot d\mathbf{r}$$
   where $\mathbf{F}(x, y, z) = \langle 6xyz, 3x^2z, 3x^2y \rangle$ and C is the path with parametrization
   $$\mathbf{r}(t) = \langle t, \sin t, t \sin t \rangle \qquad 0 \leq t \leq \pi$$
   (c) (2 points) Evaluate
   $$\oint_C \mathbf{F} \cdot d\mathbf{r}$$
   where $\mathbf{F}(x, y) = \langle xy^2, x^2 - y \rangle$ and C is the closed square path with corners $(0, 0)$, $(0, 2)$, $(2, 2)$ and $(2, 0)$ oriented *clockwise*.

4. Let
   $$\mathbf{F}(x, y, z) = \langle x^2y, xyz, z^2 \rangle$$
   (a) (5 points) Compute
   $$\operatorname{curl} \mathbf{F}$$
   (b) (5 points) Compute
   $$\operatorname{div} \mathbf{F}$$
   (c) (5 points) Compute
   $$\operatorname{div}(\operatorname{curl} \mathbf{F})$$

5. Compute the following surface integrals using any technique you like:

(a) (5 points) Evaluate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

where $\mathbf{F}(x,y,z) = \langle z,z,z\rangle$ and S is the upper half of the sphere or radius 2 with center $(0,0,0)$ oriented *inwards*.

(b) (2 points) Evaluate

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

where $\mathbf{F}(x,y,z) = \langle x+y+z,0,0\rangle$ and S is the upper half of the sphere or radius 2 with center $(0,0,0)$ oriented oriented *inwards*.

6. (5 points) Change the order of integration for the double integral $\displaystyle\int_0^{\pi}\int_0^{\sin x} xy-2\,dy\,dx.$

7. (5 points) Evaluate the double integral

$$\iint_R \frac{2y}{1+x^2}\,dA$$

where $R = \{(x,y)\,|\,0\le x\le 1 \text{ and } 0\le y\le\sqrt{x}\}$.

8. (5 points) Compute the maximum value for the function $f(x,y) = x - yx$ on the region

$$R = \{(x,y)\,|\,x^2+y^2\le 1\}$$

9. (5 points) Compute the curvature κ of the curve $y = \cos x$ at the point $(0,1)$.

10. (5 points) Find an equation for the tangent plane to the level surface of the scalar function

$$f(x,y,z) = x^2y - z^2x + 2$$

passing though the point $(1,2,-1)$

11. Show that the following vector fields are not conservative on their implicit domains.
    (a) (5 points) $\mathbf{F}(x,y,z) = \langle x^2y, 2y, z-x\rangle$
    (b) (5 points) $\mathbf{F}(x,y) = \left\langle \dfrac{x}{x^2+y^2}, \dfrac{-y}{x^2+y^2}\right\rangle$

12. Find potential functions for the following conservative vector fields
    (a) (5 points) $\mathbf{F}(x,y) = \langle 1 - y\cos(xy), -x\cos(xy)\rangle$
    (b) (5 points) $\mathbf{F}(x,y,z) = \langle yz, xz+3, xy+2\rangle$

13. (5 points) Find the x-coordinate of the center of mass of a thin triangular plate with vertices $(0,0)$, $(0,1)$ and $(2,0)$ with density function

$$f(x,y) = x+1.$$