Practice Midterm 1 – Math 2153

1. Decide if the following statements are TRUE or FALSE and circle your answer. **You do NOT need to justify your answers.**

   (a) (1 point) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^3$ then $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = 0$.

   Solution: T

   (b) (1 point) If $\mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t) \rangle$ is a vector-valued function and $|\mathbf{r}(t)| = 4$ for all $t \in \mathbf{R}$ then $\mathbf{r} \cdot \mathbf{r}' = 0$ for all t.

   Solution: T

   (c) (1 point) Let $\mathbf{u}$ and $\mathbf{v}$ be a vectors in $\mathbf{R}^3$. Then $\mathbf{u} \cdot (\mathbf{u} - \mathbf{v})$ is a vector in $\mathbf{R}^3$.

   Solution: **F** ($\mathbf{u} \cdot (\mathbf{u} - \mathbf{v})$ is a scalar)

   (d) (1 point) Let $f = f(x, y)$ be a function of two variables which is continuous at (a, b) and $g = g(t)$ be a function which is continuous at $f(a, b)$. Let

   $$h(t) = f(g(t), g(t)).$$

   Then h is continuous at $f(a, b)$.

   Solution: **F** (f would have to be continuous at $(g(f(a, b)), g(f(a, b)))$)

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

   (a) (2 points) Give an example of a continuous vector-valued function $\mathbf{r}(t)$ which is *not* differentiable at $t = 2$.

   Solution: $\boxed{\mathbf{r}(t) = |t - 2|\mathbf{i} + t\mathbf{j}}$

   (b) (2 points) Give an example of a vector-valued function which has constant curvature $\kappa \neq 0$.

   Solution: $\boxed{\mathbf{r}(t) = \langle \cos t, \sin t \rangle}$

   (c) (2 points) Give equations for two different planes in $\mathbf{R}^3$ which are parallel to the plane $z = 2x - 5y$.

   Solution: $\boxed{2x - 5y - z = 1}$ and $\boxed{2x - 5y - z = 2}$

   (d) (2 points) Give an example of a vector $\mathbf{u} \in \mathbf{R}^2$ for which $\mathbf{u} \cdot \left\langle \cos \frac{\pi}{5}, \sin \frac{\pi}{5} \right\rangle = 0$

   Solution: $\boxed{\mathbf{u} = \mathbf{0} = \langle 0, 0 \rangle}$

3. Let $\mathbf{u} = \langle 3, 2, 2 \rangle$ and $\mathbf{v} = \langle 1, -4, 6 \rangle$.

   (a) (2 points) Compute $2\mathbf{u} - \mathbf{v}$.

Solution: $2\mathbf{u} - \mathbf{v} = 2\langle 3, 2, 2\rangle - \langle 1, -4, 6\rangle = \langle 6, 4, 4\rangle - \langle 1, -4, 6\rangle = \boxed{\langle 5, 8, -2\rangle}$

(b) (2 points) Compute $\mathbf{u} \cdot \mathbf{v}$.

Solution: $\mathbf{u} \cdot \mathbf{v} = \langle 3, 2, 2\rangle \cdot \langle 1, -4, 6\rangle = 3 \cdot 1 + 2(-4) + 2 \cdot 6 = \boxed{7}$

(c) (2 points) Compute $\mathbf{u} \times \mathbf{v}$.

Solution:

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 2 \\ 1 & -4 & 6 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 2 \\ -4 & 6 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} \mathbf{k} \\
&= (2 \cdot 6 - (-4) \cdot 2)\mathbf{i} - (3 \cdot 6 - 1 \cdot 2)\mathbf{j} + (3 \cdot (-4) - 1 \cdot 2)\mathbf{k} \\
&= \boxed{20\mathbf{i} - 16\mathbf{j} - 14\mathbf{k}}
\end{aligned}$$

(d) (2 points) Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.

Solution:

$$\begin{aligned}
\text{Angle between } \mathbf{u} \text{ and } \mathbf{v} &= \arccos\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right) \\
&= \arccos\left(\frac{\langle 3, 2, 2\rangle \cdot \langle 1, -4, 6\rangle}{|\langle 3, 2, 2\rangle|\,|\langle 1, -4, 6\rangle|}\right) \\
&= \arccos\left(\frac{7}{\sqrt{3^2 + 2^2 + 2^2}\sqrt{1^2 + (-4)^2 + 6^2}}\right) \\
&= \arccos\left(\frac{7}{\sqrt{17 \cdot 53}}\right) \\
&= \boxed{\arccos\left(\frac{7}{\sqrt{901}}\right)}
\end{aligned}$$

(e) (2 points) Compute $\mathrm{proj}_{\mathbf{u}}\mathbf{v}$

Solution:

$$\begin{aligned}
\mathrm{proj}_{\mathbf{u}}\mathbf{v} &= \frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \\
&= \frac{\langle 1, -4, 6\rangle \cdot \langle 3, 2, 2\rangle}{\langle 3, 2, 2\rangle \cdot \langle 3, 2, 2\rangle}\langle 3, 2, 2\rangle \\
&= \frac{1 \cdot 3 - 4 \cdot 2 + 6 \cdot 2}{3 \cdot 3 + 2 \cdot 2 + 2 \cdot 2}\langle 3, 2, 2\rangle
\end{aligned}$$

$$= \frac{7}{17}\langle 3, 2, 2\rangle$$
$$= \boxed{\left\langle \frac{21}{17}, \frac{14}{17}, \frac{14}{17} \right\rangle}$$

4. A ball is thrown from an intial height of $20\,\mathrm{m}$ above with an initial speed of $10\,\mathrm{m/s}$. and initial angle of $\frac{\pi}{3}$ radians with the ground.
   (a) (3 points) Give the position vector of the ball $\mathbf{r}(t)$ at time t.

Solution: The acceleration of gravity is constant so at time t the acceleration of the ball is

$$\mathbf{a}(t) = \langle 0, -g\rangle = \langle 0, -9.8\rangle.$$

Let $\mathbf{v}_0$ be the initial velocity. Since the initial speed is 10 and the ball is thrown at angle $\frac{\pi}{3}$ we have

$$\mathbf{v}_0 = 10\left\langle \cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right\rangle = 10\left\langle \frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = \left\langle 5, 5\sqrt{3}\right\rangle.$$

Thus at time t the velocity of the ball is

$$\begin{aligned}\mathbf{v}(t) &= \int_0^t \mathbf{a}(w)\ \mathrm{d}w + \mathbf{v}_0\\ &= \int_0^t \langle 0, -9.8\rangle\ \mathrm{d}w + \left\langle 5, 5\sqrt{3}\right\rangle\\ &= \langle 0, -9.8w\rangle\Big|_0^t + \left\langle 5, 5\sqrt{3}\right\rangle\\ &= \langle 0, -9.8t\rangle + \left\langle 5, 5\sqrt{3}\right\rangle\\ &= \langle 5, -9.8t + 5\sqrt{3}\rangle\end{aligned}$$

Let $\mathbf{r}_0$ be the initial position. Since the ball starts at height 20 we have

$$\mathbf{r}_0 = \langle 0, 20\rangle.$$

Thus at time t the position of the ball is

$$\begin{aligned}\mathbf{r}(t) &= \int_0^t \mathbf{v}(w)\ \mathrm{d}w + \mathbf{r}_0\\ &= \int_0^t \langle 5, -9.8w + 5\sqrt{3}\rangle\ \mathrm{d}w + \langle 0, 20\rangle\\ &= \left\langle 5w, \frac{-9.8}{2}w^2 + 5\sqrt{3}w\right\rangle\Big|_0^t + \langle 0, 20\rangle\\ &= \left\langle 5t, \frac{-9.8}{2}t^2 + 5\sqrt{3}t\right\rangle + \langle 0, 20\rangle\\ &= \boxed{\left\langle 5t, -4.9t^2 + 5\sqrt{3}t + 20\right\rangle}\end{aligned}$$

(b) (2 points) Set up but do not evaluate an integral which gives the length of the path traveled by the ball. This is **not** the distance traveled by the ball along the ground, but the length of the path that the ball traces in the air.

Solution: First we need to calculate the amount of time T that the ball is in the air. At time T the ball hits the ground so the height will be 0. Thus we need to solve for T in the equation:

$$-4.9T^2 + 5\sqrt{3}T + 20 = 0$$

Using the quadratic equation we get that

$$\begin{aligned} T &= \frac{-5\sqrt{3} \pm \sqrt{(5\sqrt{3})^2 - 4(-4.9)20}}{2(-4.9)} \\ &= \frac{-5\sqrt{3} \pm \sqrt{25 \cdot 3 + 80 \cdot 4.9}}{-9.8} \\ &= \frac{5\sqrt{3} \pm \sqrt{75 + 8 \cdot 49}}{9.8} \end{aligned}$$

We want the positive root so

$$T = \frac{5\sqrt{3} + \sqrt{75 + 392}}{9.8} = \frac{\sqrt{75} + \sqrt{467}}{9.8}$$

Therefore the length of the path travelled by the ball is

$$\begin{aligned} L &= \int_0^T |\mathbf{v}(t)| \, \mathsf{d}t \\ &= \int_0^T \left|\langle 5, -9.8t + 5\sqrt{3}\rangle\right| \, \mathsf{d}t \\ &= \int_0^T \sqrt{5^2 + (-9.8t + 5\sqrt{3})^2} \, \mathsf{d}t \\ &= \boxed{\int_0^{\frac{\sqrt{75}+\sqrt{467}}{9.8}} \sqrt{5^2 + (-9.8t + 5\sqrt{3})^2} \, \mathsf{d}t} \end{aligned}$$

5. (5 points) A weight of $5\,\mathrm{N}$ is tied to two strings, both fastened to the ceiling. The first string is tied to the weight at position $\langle 0, 0\rangle$ and is fastened to the ceiling at position $\langle 3, 2\rangle$ The second string is tied to the weight at position $\langle 0, 0\rangle$ and is fastened to the ceiling at position $\langle -1, 2\rangle$. Compute the force vectors which give the forces that the strings exhert on the weight.

Solution: Let $\mathbf{F}_1$ be the force applied by the first string and $\mathbf{F}_2$ be the force applied by the second string. There must be scalars a and b such that

$$\begin{aligned} \mathbf{F}_1 &= a\big(\langle 3, 2\rangle - \langle 0, 0\rangle\big) = \langle 3a, 2a\rangle \\ \mathbf{F}_2 &= b\big(\langle -1, 2\rangle - \langle 0, 0\rangle\big) = \langle -b, 2b\rangle \end{aligned}$$

Let $\mathbf{G}$ be the force of gravity on the weight. The force has magnitude $5\,\mathrm{N}$ so

$$\mathbf{G} = \langle 0, -5 \rangle$$

The weight is stationary so all of the forces must balance. Hence

$$\begin{aligned}
&\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{G} = \mathbf{0} \\
\implies &\langle 3a, 2a \rangle + \langle -b, 2b \rangle + \langle 0, -5 \rangle = \langle 0, 0 \rangle \\
\implies &\langle 3a - b, 2a + 2b - 5 \rangle = \langle 0, 0 \rangle \\
\implies &3a - b = 0 \text{ and } 2a + 2b - 5 = 0
\end{aligned}$$

Solving for b in terms of a we get $b = 3a$. Now substituting in

$$\begin{aligned}
&2a + 2(3a) - 5 = 0 \\
\implies &8a - 5 = 0 \\
\implies &8a = 5 \\
\implies &a = \tfrac{5}{8} \\
\implies &b = 3 \cdot \tfrac{5}{8} = \tfrac{15}{8}.
\end{aligned}$$

Hence the force vector applied by the first string is

$$\mathbf{F}_1 = a\langle 3, 2 \rangle = \tfrac{5}{8}\langle 3, 2 \rangle = \boxed{\left\langle \tfrac{15}{8}, \tfrac{5}{4} \right\rangle}$$

The force vector applied by the second string is

$$\mathbf{F}_2 = b\langle -1, 2 \rangle = \tfrac{15}{8}\langle -1, 2 \rangle = \boxed{\left\langle \tfrac{-15}{8}, \tfrac{15}{4} \right\rangle}$$

6. (5 points) Compute the maximum and minimum curvature for the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

Solution: Curvature is most easily calculated for vector-valued functions so we will parametrize the ellipse:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

If we let $\cos t = \frac{x}{2}$ and $\sin t = \frac{y}{3}$ then $x = 2\cos t$ and $y = 3\sin t$ so we get the parametrization

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle = \langle 2\cos t, 3\sin t \rangle \qquad t \in [0, 2\pi]$$

We can compute curvature at t using the formula $\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$ where $\mathbf{v} = \mathbf{r}'$ and $\mathbf{a} = \mathbf{r}''$.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3\cos t \rangle$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = \langle -2\cos t, -3\sin t\rangle$$

Thus

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2\sin t & 3\cos t & 0 \\ -2\cos t & -3\sin t & 0 \end{vmatrix} \\
&= \begin{vmatrix} 3\cos t & 0 \\ -3\sin t & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2\sin t & 0 \\ -2\cos t & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2\sin t & 3\cos t \\ -2\cos t & -3\sin t \end{vmatrix} \mathbf{k} \\
&= ((3\cos t)\cdot 0 - (-3\sin t)\cdot 0)\mathbf{i} - (-2\sin t \cdot 0 - (-2\cos t)\cdot 0)\mathbf{j} \\
&\quad + ((-2\sin t)(-3sint) - (-2\cos t)(3\cos t))\mathbf{k} \\
&= (6\sin^2 t + 6\cos^2 t)\mathbf{k} \\
&= 6\mathbf{k}
\end{aligned}$$

Therefore

$$|\mathbf{v} \times \mathbf{a}| = |6\mathbf{k}| = \sqrt{0^2 + 0^2 + 6^2} = 6$$

We can also compute

$$\begin{aligned}
|\mathbf{v}|^3 &= |\langle -2\sin t, 3\cos t\rangle|^3 \\
&= \left(\sqrt{(-2\sin t)^2 + (3\cos t)^2}\right)^3 \\
&= \left(4\sin^2 t + 9\cos^2 t\right)^{\frac{3}{2}}
\end{aligned}$$

Thus at time t the curvature is

$$\begin{aligned}
\kappa(t) &= \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} \\
&= \frac{6}{\left(4\sin^2 t + 9\cos^2 t\right)^{\frac{3}{2}}} \\
&= 6\left(4\sin^2 t + 9\cos^2 t\right)^{-\frac{3}{2}}
\end{aligned}$$

We want the maximum and minimum curvatures so we need the critical values of the curvature function

$$\begin{aligned}
\kappa'(t) &= 6\left(-\tfrac{3}{2}\right)\left(4\sin^2 t + 9\cos^2 t\right)^{-\frac{5}{2}}\left(4\cdot 2\sin t\cos t + 9\cdot 2\cos t(-\sin t)\right) \\
&= \frac{(-9)(-5)2\sin t\cos t}{\left(4\sin^2 t + 9\cos^2 t\right)^{\frac{5}{2}}} \\
&= \frac{45\sin 2t}{\left(4\sin^2 t + 9\cos^2 t\right)^{\frac{5}{2}}}
\end{aligned}$$

Critical values for $\kappa(t)$ will occur where $45\sin 2t = 0$ or where $4\sin^2 t + 9\cos^2 t = 0$. Since $\sin t$ and $\cos t$ are never both 0 for the same t we have $4\sin^2 t + 9\cos^2 t \neq 0$ for all t.

Therefore we need to solve for t in the equation

$$\begin{aligned} & 45\sin 2t = 0 \\ \implies & \sin 2t = 0 \\ \implies & 2t = n\pi \\ \implies & t = \frac{n\pi}{2} \end{aligned}$$

The domain of our parametrization is $t \in [0, 2\pi]$ so we need to test the 5 critical values and endpoints: $t = 0, t = \frac{\pi}{2}, t = \pi, t = \frac{3\pi}{2}, t = 2\pi$

$$\begin{aligned} \kappa(0) &= 6\left(4\sin^2 0 + 9\cos^2 0\right)^{-\frac{3}{2}} = 6 \cdot 9^{-\frac{3}{2}} = 6 \cdot 3^{-3} = \frac{6}{27} = \frac{2}{9} \\ \kappa\left(\tfrac{\pi}{2}\right) &= 6\left(4\sin^2 \tfrac{\pi}{2} + 9\cos^2 \tfrac{\pi}{2}\right)^{-\frac{3}{2}} = 6 \cdot 4^{-\frac{3}{2}} = 6 \cdot 2^{-3} = \frac{6}{8} = \frac{3}{4} \\ \kappa(\pi) &= 6\left(4\sin^2 \pi + 9\cos^2 \pi\right)^{-\frac{3}{2}} = 6 \cdot 9^{-\frac{3}{2}} = \frac{2}{9} \\ \kappa\left(\tfrac{3\pi}{2}\right) &= 6\left(4\sin^2 \tfrac{3\pi}{2} + 9\cos^2 \tfrac{3\pi}{2}\right)^{-\frac{3}{2}} = 6 \cdot 4^{-\frac{3}{2}} = \frac{3}{4} \\ \kappa(2\pi) &= 6\left(4\sin^2 2\pi + 9\cos^2 2\pi\right)^{-\frac{3}{2}} = 6 \cdot 9^{-\frac{3}{2}} = \frac{2}{9} \end{aligned}$$

Thus we can conclude the **the maximum curvature is $\frac{3}{4}$** and **the minimum curvature is $\frac{2}{9}$.**

7. (5 points) Give an equation for the plane containing the line $\mathbf{r}(t) = \langle 3 + t, 2, 1 - t\rangle$ and the point $\langle 2, -3, 1\rangle$.

Solution: First we need two vectors parallel to the plane.

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{r}(1) - \mathbf{r}(0) = \langle 3+1, 2, 1-1\rangle - \langle 3+0, 2, 1-0\rangle = \langle 1, 0, -1\rangle \\ \mathbf{v}_2 &= \langle 2, -3, 1\rangle - \mathbf{r}(0) = \langle 2, -3, 1\rangle - \langle 3+0, 2, 1-0\rangle = \langle -1, -5, 0\rangle \end{aligned}$$

A normal vector for the plane will be

$$\begin{aligned} \mathbf{n} &= \mathbf{v}_1 \times \mathbf{v}_2 \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -1 & -5 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 0 & -1 \\ -5 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -1 \\ -1 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ -1 & -5 \end{vmatrix}\mathbf{k} \\ &= (0 \cdot 0 - (-5)(-1))\mathbf{i} - (1 \cdot 0 - (-1)(-1))\mathbf{j} + (1(-5) - (-1) \cdot 0)\mathbf{k} \\ &= -5\mathbf{i} + \mathbf{j} - 5\mathbf{k} \end{aligned}$$

Thus an equation for the plane is

$$\mathbf{n} \cdot \langle x - 2, y - (-3), z - 1\rangle = 0$$

$$\begin{gathered}-5(x-2)+(y+3)-5(z-1)=0\\ -5x+10+y+3-5z+5=0\\ \boxed{-5x+y-5z=-18}\end{gathered}$$

8. (5 points) Compute the normal and tangential components of acceleration (a_N and a_T) for a particle whose position at time t is given by the vector-valued function

$$\mathbf{r}(t)=\langle t^2, t+2\rangle.$$

Solution:

$$\begin{aligned}\mathbf{v}(t)&=\mathbf{r}'(t)=\langle 2t,1\rangle\\ \mathbf{a}(t)&=\mathbf{r}''(t)=\langle 2,0\rangle\end{aligned}$$

$$\begin{aligned}a_T&=|\text{proj}_{\mathbf{v}}\mathbf{a}|\\ &=\left|\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right|\\ &=\frac{|\mathbf{a}\cdot\mathbf{v}|}{|\mathbf{v}|}\\ &=\frac{|\langle 2,0\rangle\cdot\langle 2t,1\rangle|}{|\langle 2t,1\rangle|}\\ &=\boxed{\frac{|4t|}{\sqrt{4t^2+1}}}\end{aligned}$$

$$\begin{aligned}a_N&=|a_N\mathbf{N}|\\ &=|\mathbf{a}-a_T\mathbf{T}|\\ &=\left|\mathbf{a}-\frac{\mathbf{a}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\right|\\ &=\left|\langle 2,0\rangle-\frac{\langle 2,0\rangle\cdot\langle 2t,1\rangle}{\langle 2t,1\rangle\cdot\langle 2t,1\rangle}\langle 2t,1\rangle\right|\\ &=\left|\langle 2,0\rangle-\frac{4t}{4t^2+1}\langle 2t,1\rangle\right|\\ &=\left|\langle 2,0\rangle-\left\langle\frac{8t^2}{4t^2+1},\frac{4t}{4t^2+1}\right\rangle\right|\\ &=\left|\left\langle\frac{8t^2+2}{4t^2+1}-\frac{8t^2}{4t^2+1},\frac{4t}{4t^2+1}\right\rangle\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| \left\langle \frac{2}{4t^2+1}, \frac{4t}{4t^2+1} \right\rangle \right| \\
&= \sqrt{\left(\frac{2}{4t^2+1}\right)^2 + \left(\frac{4t}{4t^2+1}\right)^2} \\
&= \sqrt{\frac{4+16t^2}{(4t^2+1)^2}} \\
&= \frac{\sqrt{4+16t^2}}{4t^2+1} \\
&= \frac{2\sqrt{1+4t^2}}{4t^2+1} \\
&= \boxed{\frac{2}{\sqrt{4t^2+1}}}
\end{aligned}
$$