# Practice Midterm 2 – Math 2153

1. Decide if the following statements are TRUE or FALSE and circle your answer. **You do NOT need to justify your answers.**

   (a) (1 point) If both partial derivatives $f_x$ and $f_y$ exist at $(a, b)$ then $f$ is differentiable at $(a, b)$.

   > **Solution: F** ($f_x$ and $f_y$ must exist and be continuous on an open set containing the point $(a, b)$ to ensure that $f$ is differentiable at $(a, b)$.)

   (b) (1 point) If $f$ has a local maximum at the point $(a, b, c)$ then $\nabla f = \mathbf{0}$.

   > **Solution: F** ($f$ can also have a local maximum at $(a, b, c)$ when $\nabla f$ does not exist.)

   (c) (1 point) If $f$ has a saddle point at $(a, b)$ then $f$ cannot have a local minimum at $(a, b)$.

   > **Solution: T** (The definition of "saddle point" precludes it from being a local maximum or minimum)

   (d) (1 point) If $f$ is differentiable at $(a, b, c)$ then magnitude of the gradient vector $\nabla f(a, b, c)$ is the maximal directional derivative

   $$D_{\mathbf{u}}(a, b, c)$$

   where $\mathbf{u}$ ranges over all unit vectors in $\mathbf{R}^3$.

   > **Solution: T**

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

   (a) (2 points) Give an example of a function $f(x, y)$ continuous on $\mathbf{R}^2$ such that there are infinitely many points $(a, b) \in \mathbf{R}^2$ such that $f$ has a local maximum at $(a, b)$.

   > **Solution:** $\boxed{f(x, y) = 0}$

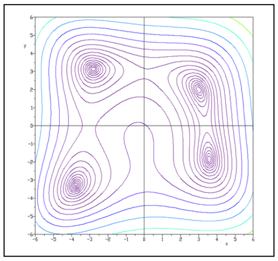
   (b) (2 points) Give an example of a function $F(x, y, z)$ for which the graph of $z = \sin(xy)$ is a level surface.

   > **Solution:** $\boxed{F(x, y, z) = z - \sin(xy)}$

   (c) (2 points) Give an example of a function $f(x, y)$ with domain $\mathbf{R}^2$ for which $f_x(0, 0)$ exists but $f_y(0, 0)$ does not exist.

   > **Solution:** $\boxed{f(x, y) = |y|}$

   (d) (2 points) Sketch level curves for a function $f(x, y)$ with four local maximuma and no local minima. Make sure to include enough level curves to illustrate these properties.

**Solution:** 

3. Compute the following:

(a) (2 points) $\frac{\partial}{\partial y}\left(\frac{\ln(xy)}{x^y}\right)$

**Solution:**

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\ln(xy)}{x^y}\right) &= \frac{\partial}{\partial y}\left(\frac{\ln(xy)}{e^{y\ln x}}\right)\\ &= \frac{\frac{ye^{y\ln x}}{yx} - \ln(x)e^{y\ln x}\ln(xy)}{e^{2y\ln x}}\\ &= \boxed{\tfrac{x^{y-1}-\ln(x)x^y\ln(xy)}{x^{2y}}}\end{aligned}$$

(b) (2 points) $D_{\mathbf{u}}(3x^3y - y)$ where $\mathbf{u} = \left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right\rangle$.

**Solution:** Let $f(x,y) = 3x^3y - y$.

$$\begin{aligned}D_{\mathbf{u}}(3x^3y - y) &= \nabla f \cdot \mathbf{u}\\ &= \langle 9x^2y, 3x^3 - 1\rangle \cdot \left\langle \frac{-2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right\rangle\\ &= \boxed{\tfrac{-18}{\sqrt{5}}x^2y - \tfrac{-1}{\sqrt{5}}(3x^3-1)}\end{aligned}$$

(c) (2 points) Find the unit vector $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ pointing in the direction of maximum increase for the function $f(x,y,z) = xyz^2$ at the point $(1,1,1)$.

**Solution:**

$$\begin{aligned}\nabla f &= \langle yz^2, xz^2, 2xyz\rangle\\ \nabla f(1,1,1) &= \langle 1,1,2\rangle\\ \mathbf{u} &= \frac{\nabla f(1,1,1)}{|\nabla f(1,1,1)|}\\ &= \frac{\langle 1,1,2\rangle}{|\langle 1,1,2\rangle|}\\ &= \boxed{\left\langle \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{2}{\sqrt{6}}\right\rangle}\end{aligned}$$

(d) (2 points) $\int_1^2 \int_0^\pi \sin(x+y)\,dx\,dy$.

**Solution:**

$$\begin{aligned}\int_1^2 \int_0^\pi \sin(x+y)\,dx\,dy &= \int_1^2 -\cos(x+y)\Big|_{x=0}^{\pi} dy \\ &= \int_1^2 -\cos(\pi+y)+\cos y\,dy \\ &= \int_1^2 \cos y+\cos y\,dy \\ &= 2\sin y\Big|_{x=1}^{2} \\ &= \boxed{2\sin 2 - 2\sin 1}\end{aligned}$$

(e) (2 points) $\int_1^2 \int_0^2 \int_0^1 \frac{xz}{y}\,dz\,dx\,dy$.

**Solution:**

$$\begin{aligned}\int_1^2 \int_0^2 \int_0^1 \frac{xz}{y}\,dz\,dx\,dy &= \int_1^2 \int_0^2 \frac{xz^2}{2y}\Big|_{z=0}^{1} dx\,dy \\ &= \int_1^2 \int_0^2 \frac{x}{2y}\,dx\,dy \\ &= \int_1^2 \frac{x^2}{4y}\Big|_{x=0}^{2} dy \\ &= \int_1^2 \frac{1}{y}\,dy \\ &= \ln y\Big|_{y=1}^{2} \\ &= \ln 2 - \ln 1 \\ &= \boxed{\ln 2}\end{aligned}$$

(f) (2 points) Compute $\frac{\partial z}{\partial x}$ in terms of $x$, $y$ and $z$ if $z$ satisfies the implicit equation

$$xy + yz + xz = 7$$

**Solution:**

$$\begin{aligned}\frac{\partial}{\partial x}(xy+yz+xz) &= \frac{\partial}{\partial x}(0) \\ y + y\frac{\partial z}{\partial x} + z + x\frac{\partial z}{\partial x} &= 0 \\ y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial x} &= -y - z \\ \frac{\partial z}{\partial x}(y+x) &= -y - z\end{aligned}$$

$$\frac{\partial z}{\partial x} = \boxed{\frac{-y-z}{y+x}}$$

4. Change the order of integration for the following double integrals. You may have to express the new integral as a sum of double integrals. **You do not need to evaluate the integrals.**

(a) (2 points) $\int_1^2 \int_{x-1}^{\ln x} xy^2\,dy\,dx.$

**Solution:**

$$\begin{aligned}\int_1^2 \int_{x-1}^{\ln x} xy^2\,dy\,dx &= \int_1^2 \int_{\ln x}^{x-1} -xy^2\,dy\,dx \\ &= \int_0^{\ln 2} \int_{y+1}^{e^y} -xy^2\,dx\,dy + \int_{\ln 2}^{1} \int_{y+1}^{2} -xy^2\,dx\,dy \\ &= \boxed{\int_0^{\ln 2} \int_{e^y}^{y+1} xy^2\,dx\,dy + \int_{\ln 2}^{1} \int_{y+1}^{2} xy^2\,dx\,dy}\end{aligned}$$

(b) (2 points) $\int_0^{\pi/4} \int_{\sin y}^{\cos y} xy^2\,dx\,dy.$

**Solution:**

$$\int_0^{\pi/4} \int_{\sin y}^{\cos y} xy^2\,dx\,dy = \boxed{\int_0^{\sqrt{2}/2} \int_0^{\arcsin x} xy^2\,dy\,dx + \int_{\sqrt{2}/2}^{1} \int_0^{\arccos x} xy^2\,dy\,dx}$$

5. (5 points) Give a triple integral which computes the volume of the bounded region in $\mathbf{R}^3$ enclosed by the surfaces

$$x^2 + y^2 - z^2 = 9, \qquad z = 4, \qquad z = -4$$

**Solution:** Let $D$ be the region in $\mathbf{R}^3$ described above. Then

$$\begin{aligned}\text{Volume of } D &= \iiint_D dV \\ &= \boxed{\int_{-4}^{4} \int_{-\sqrt{9+z^2}}^{\sqrt{9+z^2}} \int_{-\sqrt{9+z^2-x^2}}^{\sqrt{9+z^2-x^2}} dy\,dx\,dz}\end{aligned}$$

6. (5 points) Compute the maximum and minimum values for the function $f(x, y) = x - y^2 + xy$ on the region in the plane bounded by the ellipse $y^2 + 9x^2 = 9$.

**Solution:** First we find the critical points of $f$.

$$\nabla f = \langle 1+y, -2y+x \rangle$$

Thus $0 = 1+y$ and $0 = -2y+x$. Hence $y = -1$ and $0 = (-2)(-1)+x$. Thus the critical point of $f$ is $(-2,-1)$. This point is not in the region on which we are maximizing since $(-1)^2 + 9(-2)^2 > 9$. Thus there are no critical points in the interior of the ellipse.

Now let $g(x,y) = y^2 + 9x^2 - 9$. We wish to maximize $f$ subject to the constraint $g(x,y) = 0$. Applying the method of Lagrange multipliers we compute

$$\nabla g = \langle 18x, 2y \rangle$$

Setting $\nabla f = \lambda \nabla g$ gives the two equations

$$\begin{aligned} 1+y &= \lambda 18x \\ -2y+x &= \lambda 2y \end{aligned}$$

Solving the second equation for $\lambda$ we get $\lambda = \frac{-2y+x}{2y}$. Substituting this into the first equation we get

$$\begin{aligned} 1+y &= \left(\frac{-2y+x}{2y}\right) 18x \\ 2y+2y^2 &= -36xy + 18x^2 \\ 2y+2y^2+36xy-18x^2 &= 0 \end{aligned}$$

From the constraint equation we get $y^2 = 9 - 9x^2$ so

$$\begin{aligned} 2y + 2(9-9x^2) + 36xy - 18x^2 &= 0 \\ 2y + 18 + 36xy - 36x^2 &= 0 \\ (2+36x)y &= 36x^2 - 18 \\ y &= \frac{9(2x^2-1)}{1+18x} \\ y^2 &= \frac{81(4x^4+4x^2-1)}{1+36x+324x^2} \\ 9-9x^2 &= \frac{81(4x^4+4x^2-1)}{1+36x+324x^2} \end{aligned}$$

You might have noticed by now that computing $x$ here is not a reasonable task ...

7. (5 points) Use the method of Lagrange multipliers to find the maximum volume for a lidless can with surface area 1.

**Solution:** The volume of a can with radius $r$ and height $h$ is

$$V(r,h) = \pi r^2 h.$$

The surface area for a lidless can with radius $r$ and height $h$ is

$$S(r,h) = \pi r^2 + 2\pi r h.$$

We wish to maximize $V$ subject to the constraint that $S(r,h) = 1$

$$\begin{aligned} \nabla V =& \langle 2\pi rh, \pi r^2 \rangle \\ \nabla S =& \langle 2\pi r + 2\pi h, 2\pi r \rangle \end{aligned}$$

The vector equation $\nabla V = \lambda \nabla S$ gives the two equations

$$\begin{aligned} 2\pi rh &= \lambda(2\pi r + 2\pi h) \\ \pi r^2 &= \lambda 2\pi r \end{aligned}$$

Solving for $\lambda$ in the second equation we get

$$\lambda = \frac{r}{2}$$

Substituting $\lambda = \frac{r}{2}$ into the first equation and cancelling $\pi$ we get

$$2rh = \frac{r}{2}(2r + 2h)$$

and hence

$$\begin{aligned} 2rh &= r^2 + rh \\ rh &= r^2 \\ h &= r \end{aligned}$$

Substituting $h = r$ back into our constraint $S(r,h) = 1$ gives

$$\begin{aligned} 1 = \pi r^2 + 2\pi r^2 \\ 1 = 3\pi r^2 \\ \pm\frac{1}{\sqrt{3\pi}} = r \end{aligned}$$

Radii must be positive so only the positive root makes sense. We have $h = r$ so the maximum volume must occur at $(r,h) = \left(\frac{1}{\sqrt{3\pi}}, \frac{1}{\sqrt{3\pi}}\right)$. Thus the maximum volume is

$$\begin{aligned} V\left(\frac{1}{\sqrt{3\pi}}, \frac{1}{\sqrt{3\pi}}\right) &= \pi\left(\frac{1}{\sqrt{3\pi}}\right)^2\left(\frac{1}{\sqrt{3\pi}}\right) \\ &= \boxed{\frac{1}{3\sqrt{3\pi}}} \end{aligned}$$

8. (5 points) Estimate the change in $z = xy^2 - x^2 + y$ when $(x,y)$ changes from $(1,2)$ to $(1.1, 1.9)$.

**Solution:** Let $f(x,y) = xy^2 - x^2 + y$. Then

$$\begin{aligned} f_x &= y^2 - 2x \\ f_y &= 2xy + 1 \end{aligned}$$

so

$$f_x(1,2) = 2^2 - 2(1) = 2$$
$$f_y(1,2) = 2(1)(2) + 1 = 5.$$

Thus we can estimate the change $\Delta z$ in $z$ using the differential $dz$

$$\begin{aligned}
\Delta z &\approx dz \\
&= f_x(1,2)\,dx + f_y(1,2)\,dy \\
&= f_x(1,2) \cdot (1.1 - 1) + f_y(1,2) \cdot (1.9 - 2) \\
&= 2 \cdot (0.1) + 5 \cdot (-0.1) \\
&= \boxed{-0.3}
\end{aligned}$$