Practice Final Solutions – Math 2255

** **Practice midterms I and II** provide good practice problems for previous material.
** **Bring TWO double-sided** 8.5×11 **sheets of notes to use during the final**.

1. Decide if the following statements are TRUE or FALSE. **You do NOT need to justify your answers.**

   (a) (2 points) Let L_1 and L_2 be linear second order linear differential operators. If $f(t)$ is a solution to the differential equations

   $$L_1[y] = 0$$

   and $g(t)$ is a solution to the differential equation

   $$L_2[y] = 0$$

   then $f(t) + g(t)$ is a solution to the differential equation

   $$(L_1 + L_2)[y] = 0.$$

Solution: F
If $L_1[f(t)] = 0$ and $L_2[g(t)] = 0$ then

$$\begin{aligned}(L_1 + L_2)[f(t) + g(t)] &= (L_1 + L_2)[f(t)] + (L_1 + L_2)[g(t)] \\ &= L_1[f(t)] + L_2[f(t)] + L_1[g(t)] + L_2[g(t)] \\ &= 0 + L_2[f(t)] + L_1[g(t)] + 0 \\ &= L_2[f(t)] + L_1[g(t)].\end{aligned}$$

but in general $L_2[f(t)] + L_1[g(t)]$ will be nonzero.

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

   (a) (2 points) Give an example of a linear differential equation for which $x_0 = 4$ is a regular singular point.

Solution: $\boxed{(x-4)^2 y'' + y = 0}$

   (b) (2 points) Give an example of a linear differential equation for which $x_0 = 4$ is a irregular singular point.

Solution: $\boxed{(x-4)^3 y'' + y = 0}$

3. (10 points) Compute the radius of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{n^2 6^n (x-4)^n}{(n+1)8^{n+2}}$$

Solution:

$$\lim_{n\to\infty} \frac{\left|\frac{(n+1)^2 6^{n+1}(x-4)^{n+1}}{(n+2)8^{n+3}}\right|}{\left|\frac{n^2 6^n (x-4)^n}{(n+1)8^{n+2}}\right|} = \lim_{n\to\infty} \left|\frac{(n+1)^2 6^{n+1}(x-4)^{n+1} \cdot (n+1)8^{n+2}}{(n+2)8^{n+3} \cdot n^2 6^n (x-4)^n}\right|$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left|\frac{(n+1)^3\cdot 6(x-4)}{n^2(n+2)8}\right| \\
&= \lim_{n\to\infty}\left|\frac{n^{-3}}{n^{-3}}\cdot\frac{(n+1)^3\cdot 6(x-4)}{n^2(n+2)8}\right| \\
&= \lim_{n\to\infty}\left|\frac{(1+n^{-1})^3\cdot 6(x-4)}{1^2(1+2n^{-1})8}\right| \\
&= \tfrac{3}{4}\left|x-4\right|
\end{aligned}
$$

Applying the Ratio Test we see that if $|x-4| < \frac{4}{3}$ then the series converges absolutely and if $|x-4| > \frac{4}{3}$ then the series diverges. Thus the radius of convergence is $\boxed{\frac{4}{3}}$.

4. (10 points) Find the general solution to the differential equation

$$x^2y'' - 3xy' + 3y = x^4.$$

Solution: First we want a fundamental set of solutions to the homogeneous Euler equation

$$x^2y'' - 3xy' + 3y = 0$$

which has indicial equation

$$
\begin{aligned}
r(r-1) - 3r + 3 &= 0 \\
r^2 - r - 3r + 3 &= 0 \\
r^2 - 4r + 3 &= 0 \\
(r-1)(r-3) &= 0.
\end{aligned}
$$

The roots are therefore $r_1 = 3$ and $r_2 = 1$. Thus a fundamental set of solutions to the homogeneous Euler equation is $y_1 = |x|^3$ and $y_2 = |x|$. If $x > 0$ then the Wronskian of these solutions is

$$
\begin{aligned}
W(y_1, y_2)(x) &= \begin{vmatrix} x^3 & x \\ 3x^2 & 1 \end{vmatrix} \\
&= x^3 - 3x^3 \\
&= -2x^3
\end{aligned}
$$

In standard form the differential equation is

$$y'' - \tfrac{3y'}{x} + \tfrac{3y}{x^2} = x^2.$$

Hence for $x > 0$ we may apply variation of parameters to get particular solution

$$
\begin{aligned}
Y(x) &= -y_1(x)\int_1^x \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\,\mathrm{d}s + y_2(x)\int_1^x \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,\mathrm{d}s \\
&= -x^3\int_1^x \frac{s\cdot s^2}{-2s^3}\,\mathrm{d}s + x\int_1^x \frac{s^3s^2}{-2s^3}\,\mathrm{d}s \\
&= -x^3\int_1^x -\tfrac{1}{2}\,\mathrm{d}s + x\int_1^x -\tfrac{1}{2}s^2\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&= -x^3\left(-\tfrac{s}{2}\right)\Big|_{s=1}^{x} + x\left(-\tfrac{s^3}{6}\right)\Big|_{s=1}^{x} \\
&= -x^3\left(-\tfrac{x}{2}+\tfrac{1}{2}\right) + x\left(-\tfrac{x^3}{6}+\tfrac{1}{6}\right) \\
&= \tfrac{x^4}{2} - \tfrac{x^3}{2} - \tfrac{x^4}{6} + \tfrac{x}{6} \\
&= \tfrac{x^4}{3} - \tfrac{x^3}{2} + \tfrac{x}{6}.
\end{aligned}
$$

The general solution of the original nonhomogeneous equation is therefore

$$
\begin{aligned}
y(x) &= Y(x) + C_1x^3 + C_2x \\
&= \tfrac{x^4}{3} - \tfrac{x^3}{2} + \tfrac{x}{6} + C_1x^3 + C_2x \\
&= \boxed{\tfrac{x^4}{3} + K_1x^3 + K_2x}.
\end{aligned}
$$

for $x > 0$. A similar calculation gives the same general solution for $x < 0$. One can verify that this general solution works for $x = 0$ as well. Thus the solution above works for all $x \in \mathbf{R}$.

5. (10 points) Give a positive lower bound with justification for the radius of convergence for any series solution to the differential equation

$$y'' - \frac{y'}{e - e^x} + \frac{y}{4x^2+1} = 0$$

centered at $x = -1$.

Solution: The roots of $e - e^x$ are $x = 1 + 2\pi ni$ for all integers $n \in \mathbf{Z}$. The closest such root to -1 is when $n = 0$ so the Taylor Series for $\frac{1}{e-e^x}$ centered at $x = -1$ has radius of convergence $R_1 = |1-(-1)| = 2$.

The roots of $4x^2+1$ are $x = \frac{\pm\sqrt{-16}}{2\cdot 4} = \pm\frac{i}{2}$. Both of these roots are at a distance $\sqrt{(-1)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2}$ from -1. Thus the Taylor Series for $\frac{1}{4x^2+1}$ centered at $x = -1$ has radius of convergence $R_2 = \frac{\sqrt{5}}{2}$.

Therefore the radius of convergence for a series solution to the differential equation centered at $x = -1$ must be at least $\min\left\{2, \frac{\sqrt{5}}{2}\right\} = \boxed{\frac{\sqrt{5}}{2}}$.

6. (10 points) Find the general series solution for the differential equation

$$y'' + x^3y = 0$$

centered at $x = 0$.

Solution: 0 is an ordinary point of this differential equation so we will look for a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

Thus

$$\begin{aligned}
\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x^3 \sum_{n=0}^{\infty} a_n x^n &= 0 \\
\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+3} &= 0 \\
\sum_{n=-3}^{\infty} a_{n+5}(n+5)(n+4)x^{n+3} + \sum_{n=0}^{\infty} a_n x^{n+3} &= 0 \\
a_2 \cdot 2 \cdot 1 + a_3 \cdot 3 \cdot 2x + a_4 \cdot 4 \cdot 3x^2 + \sum_{n=0}^{\infty} a_{n+5}(n+5)(n+4)x^{n+3} + \sum_{n=0}^{\infty} a_n x^{n+3} &= 0 \\
a_2 \cdot 2 \cdot 1 + a_3 \cdot 3 \cdot 2x + a_4 \cdot 4 \cdot 3x^2 + \sum_{n=0}^{\infty} \left[a_{n+5}(n+5)(n+4) + a_n \right] x^{n+3} &= 0
\end{aligned}$$

Constant term gives equation:

$$2a_2 = 0$$

Coefficient of x gives equation:

$$6a_3 = 0$$

Coefficient of x^2 gives equation:

$$12a_4 = 0$$

Thus $a_2 = a_3 = a_4 = 0$. For $n \geq 0$ we have the recursive equation:

$$\begin{aligned}
a_{n+5}(n+5)(n+4) + a_n &= 0 \\
a_{n+5} &= \frac{-a_n}{(n+5)(n+4)}
\end{aligned}$$

For solution $y_1(x)$ we set $a_0 = 1$ and $a_1 = 0$. Then

$$\begin{aligned}
a_5 &= \tfrac{-a_0}{(5)(4)} = -\tfrac{1}{5 \cdot 4} \\
a_6 &= \tfrac{-a_1}{(6)(5)} = -\tfrac{0}{6 \cdot 5} = 0 \\
a_7 &= \tfrac{-a_2}{(7)(6)} = -\tfrac{0}{7 \cdot 6} = 0 \\
a_8 &= \tfrac{-a_3}{(8)(7)} = -\tfrac{0}{8 \cdot 7} = 0 \\
a_9 &= \tfrac{-a_4}{(9)(8)} = -\tfrac{0}{9 \cdot 8} = 0 \\
a_{10} &= \tfrac{-a_5}{(10)(9)} = \tfrac{1}{10 \cdot 9 \cdot 5 \cdot 4}
\end{aligned}$$

$$y_1(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{5n}}{5n \cdot (5n-1) \cdot 5(n-1) \cdot (5(n-1)-1) \cdot \cdots \cdot 5 \cdot 4}$$

For solution $y_2(x)$ we set $a_0 = 0$ and $a_1 = 1$. Then

$$\begin{aligned}
a_5 &= \tfrac{-a_0}{(5)(4)} = -\tfrac{0}{5\cdot 4} = 0\\
a_6 &= \tfrac{-a_1}{(6)(5)} = -\tfrac{1}{6\cdot 5}\\
a_7 &= \tfrac{-a_2}{(7)(6)} = -\tfrac{0}{7\cdot 6} = 0\\
a_8 &= \tfrac{-a_3}{(8)(7)} = -\tfrac{0}{8\cdot 7} = 0\\
a_9 &= \tfrac{-a_4}{(9)(8)} = -\tfrac{0}{9\cdot 8} = 0\\
a_{10} &= \tfrac{-a_5}{(10)(9)} = \tfrac{0}{10\cdot 9} = 0\\
a_{11} &= \tfrac{-a_6}{(11)(10)} = \tfrac{1}{11\cdot 10\cdot 6\cdot 5}
\end{aligned}$$

$$y_2(x) = x + \sum_{n=1}^{\infty} \frac{(-1)^n x^{5n+1}}{(5n+1)\cdot 5n\cdot (5(n-1)+1)\cdot 5(n-1)\cdot \cdots \cdot 6\cdot 5}$$

The general solution is therefore

$$y(x) = \boxed{a_0\left(1 + \sum_{n=1}^{\infty} \tfrac{(-1)^n x^{5n}}{5n\cdot(5n-1)\cdot 5(n-1)\cdot(5(n-1)-1)\cdots\cdot 5\cdot 4}\right) + a_1\left(x + \sum_{n=1}^{\infty} \tfrac{(-1)^n x^{5n+1}}{(5n+1)\cdot 5n\cdot(5(n-1)+1)\cdot 5(n-1)\cdots\cdot 6\cdot 5}\right)}.$$

7. (10 points) Find a nonzero series solution to the equation

$$x^2y'' + xy' + (-1+x)y = 0$$

centered at $x_0 = 0$.

Solution:

$$\lim_{x\to 0} \frac{x}{x^2} = \lim_{x\to 0} \frac{1}{x} \text{ does not exist.}$$

Thus 0 is a singular point.

$$\begin{aligned}
p_0 &= \lim_{x\to 0} \frac{x}{x^2}\cdot (x-0) = \lim_{x\to 0} 1 = 1\\
q_0 &= \lim_{x\to 0} \frac{-1+x}{x^2}\cdot (x-0)^2 = \lim_{x\to 0} -1+x = -1
\end{aligned}$$

Thus 0 is a regular singular point with indicial equation

$$F(r) = r(r-1) + r - 1 = (r+1)(r-1) = 0$$

Thus the exponents at the singularity are $r_1 = 1$ and $r_2 = -1$.
We will therefore look for a solution of the form

$$y(x) = x^1 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y'(x) = \sum_{n=0}^{\infty} a_n(n+1)x^n$$
$$y''(x) = \sum_{n=1}^{\infty} a_n(n+1)nx^{n-1}$$

Thus

$$\begin{aligned}
x^2\sum_{n=1}^{\infty} a_n(n+1)nx^{n-1} + x\sum_{n=0}^{\infty} a_n(n+1)x^n + (-1+x)\sum_{n=0}^{\infty} a_n x^{n+1} &= 0 \\
\sum_{n=1}^{\infty} a_n(n+1)nx^{n+1} + \sum_{n=0}^{\infty} a_n(n+1)x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+2} &= 0 \\
\sum_{n=1}^{\infty} a_n(n+1)nx^{n+1} + \sum_{n=0}^{\infty} a_n(n+1)x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+1} &= 0 \\
a_0 \cdot 1 \cdot x - a_0 \cdot x + \sum_{n=1}^{\infty} a_n(n+1)nx^{n+1} + \sum_{n=1}^{\infty} a_n(n+1)x^{n+1} - \sum_{n=1}^{\infty} a_n x^{n+1} + \sum_{n=1}^{\infty} a_{n-1} x^{n+1} &= 0 \\
\sum_{n=1}^{\infty} \Big[a_n(n+1)n + a_n(n+1) - a_n + a_{n-1}\Big]\, x^{n+1} &= 0
\end{aligned}$$

For all $n \geq 1$ we get the recursive equation

$$\begin{aligned}
a_n(n+1)n + a_n(n+1) - a_n + a_{n-1} &= 0 \\
a_n(n+2)n &= -a_{n-1} \\
a_n &= \frac{-a_{n-1}}{n(n+2)}
\end{aligned}$$

For the solution $y_1(x)$ we set $a_0 = 1$.

$$\begin{aligned}
a_1 &= \tfrac{-a_0}{1\cdot 3} = \tfrac{-1}{1\cdot 3} \\
a_2 &= \tfrac{-a_1}{2\cdot 4} = \tfrac{1}{1\cdot 2\cdot 3\cdot 4} \\
a_3 &= \tfrac{-a_2}{3\cdot 5} = \tfrac{-1}{1\cdot 2\cdot 3^2\cdot 4\cdot 5} \\
a_4 &= \tfrac{-a_3}{4\cdot 6} = \tfrac{1}{1\cdot 2\cdot 3^2\cdot 4^2\cdot 5\cdot 6} \\
&\vdots \\
a_n &= \tfrac{2(-1)^n}{n!(n+2)!}
\end{aligned}$$

Note that this expression works for a_0 as well. Therefore a nonzero solution is

$$y_1(x) = \boxed{\sum_{n=0}^{\infty} \frac{2(-1)^n x^{n+1}}{n!(n+2)!}}$$

8. (10 points) What is the indicial equation for the differential equation

$$x^2y'' + \frac{(\cos x - 1)y'}{x} + \frac{(\sin x)y}{3x} = 0$$

at $x = 0$.

Solution:

$$\begin{aligned} p_0 &= \lim_{x\to 0} \frac{\cos x - 1}{x \cdot x^2} \cdot (x - 0) \\ &= \lim_{x\to 0} \frac{\cos x - 1}{x^2} \\ &= \lim_{x\to 0} \frac{-\sin x}{2x} \\ &= \lim_{x\to 0} \frac{-\cos x}{2} \\ &= \frac{-\cos 0}{2} \\ &= -\tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} q_0 &= \lim_{x\to 0} \frac{\sin x}{3x \cdot x^2} \cdot (x - 0)^2 \\ &= \lim_{x\to 0} \frac{\sin x}{3x} \\ &= \lim_{x\to 0} \frac{\cos x}{3} \\ &= \frac{\cos 0}{3} \\ &= \tfrac{1}{3} \end{aligned}$$

Thus the indicial equation is

$$\boxed{r(r-1) - \tfrac{1}{2}r + \tfrac{1}{3} = 0}.$$