Practice Midterm 1 – Math 2255

*****Bring a two sided 8.5×11 sheet of notes (definitions, theorems and formulas are ok but no worked examples) to use during Midterm 1.**

1. Decide if the following statements are TRUE or FALSE. **You do NOT need to justify your answers.**

   (a) (1 point) If the functions φ and ψ are solutions to the differential equation

   $$y'' + ay = 0$$

   then $\varphi - 2\psi$ is a solution to the differential equation as well.

   Solution: T (The set of solutions to linear homogeneous differential equations form a vector space.)

   (b) (1 point) If either of the functions p or g is not differentiable at $t = t_0$ then the initial value problem

   $$y' + p(t)y = g(t), \qquad y(t_0) = y_0$$

   cannot have a unique solution on any open interval I containing t_0.

   Solution: F (The initial value problem will have a unique solution as long as p and g are continuous. For example, set t_0, $p(t) = |t|$, $g(t) = |t|$)

   (c) (1 point) It is possible for $\varphi(t) = e^t$ and $\psi(t) = 1$ to be solutions to the same first order differential equation

   $$y' = f(y, t)$$

   where f and $\frac{\partial f}{\partial y}$ are continuous on the entire (y, t)-plane.

   Solution: F (This would violate the uniqueness of the solution with initial value $y(0) = 1$)

   (d) (1 point) Let p and q be continuous on $\mathbf{R}$. Let y_1 and y_2 be solutions to the differential equation

   $$y'' + p(t)y' + q(t)y = 0.$$

   If the Wronskian determinant $W(y_1, y_2)(t_0)$ is 0 at t_0 then

   $$W(y_1, y_2)(t) = 0$$

   for all $t \in \mathbf{R}$.

   Solution: T (On an interval of continuity of p and q the Wronskian of two solutions is either always 0 or never 0.)

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

   (a) (2 points) Give an example of a first order differential equation which is separable but not linear.

   Solution: $\boxed{\frac{dy}{dx}y^2 = 1 + x}$

   (b) (2 points) Give an example of a nonhomogeneous 3rd order linear differential equation with **constant** coefficients.

Solution: $\boxed{y''' + y' - y = 3}$

(c) (2 points) Give an example of a first order autonomous differential equation with exactly two equilibrium solutions.

Solution: $\boxed{\frac{dy}{dt} = (y-2)(y+3)}$

3. (5 points) Compute the Wronskian for $y_1(t) = t^2$, $y_2(t) = \frac{1}{t}$

Solution:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} t^2 & \frac{1}{t} \\ 2t & \frac{-1}{t^2} \end{vmatrix} \\ &= t^2 \cdot \left(\frac{-1}{t^2}\right) - 2t \cdot \left(\frac{1}{t}\right) \\ &= -1 - 2 \\ &= \boxed{-3} \end{aligned}$$

4. (5 points) Find the smallest vector space of functions closed under differentiation which contains the function

$$f(t) = t^2 \sin 2t.$$

Solution: $\boxed{V = \{(At^2 + Bt + C)\sin 2t + (Et^2 + Ft + G)\cos 2t \,|\, A, B, C, E, F, G \in \mathbf{R}\}}$

5. (5 points) Find the integrating factor for the first order linear equation

$$ty' - 6y - t^2 + 1 = 0.$$

Solution: In standard format this equation is

$$y' - \frac{6}{t} \cdot y = \frac{t^2 - 1}{t}$$

Thus for $t > 0$ an integrating factor is

$$\begin{aligned} \mu(t) &= e^{\int_1^t -\frac{6}{s}\, \mathrm{d}s} \\ &= e^{-6 \ln t} \\ &= \boxed{t^{-6}.} \end{aligned}$$

For $t < 0$ an integrating factor is

$$\begin{aligned}\mu(t) &= e^{\int_{-1}^{t} -\frac{6}{s}\,\mathrm{d}s} \\ &= e^{-6\ln(-t)} \\ &= (-t)^{-6} \\ &= \boxed{t^{-6}.}\end{aligned}$$

6. Find general solutions for the following differential equations

(a) (5 points) $y' = (1+y^2)e^x$.

Solution: This is a first order nonlinear separable equation.

$$\begin{aligned}\frac{dy}{dx} &= (1+y^2)e^x \\ \frac{1}{1+y^2}\cdot\frac{dy}{dx} &= e^x \\ \int \frac{1}{1+y^2}\cdot\frac{dy}{dx}\,dx &= \int e^x dx \\ \int \frac{1}{1+y^2}\cdot dy &= \int e^x dx \\ \arctan y &= e^x + C \\ y &= \boxed{\tan(e^x + C)}\end{aligned}$$

(b) (5 points) $y'' - 4y' + 4y = 0$.

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$\begin{aligned}r^2 - 4r + 4 &= 0 \\ (r-2)^2 &= 0.\end{aligned}$$

Thus $r = 2$ is a double root.

The general solution is therefore

$$\boxed{y = C_1e^{2t} + C_2te^{2t}.}$$

(c) (5 points) $y'' - 5y' - 6y = 0$.

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$r^2 - 5r - 6 = 0.$$

which has roots

$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm \sqrt{49}}{2} = -1 \text{ or } 6$$

The general solution is therefore

$$\boxed{y = C_1 e^{-t} + C_2 e^{6t}.}$$

(d) (5 points) $y'' - 4y' + 13y = 0$.

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$r^2 - 4r + 13 = 0.$$

which has roots

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

The general solution is therefore

$$\boxed{y = C_1 e^{2t} \cos 3t + C_2 e^{2t} \sin 3t.}$$

(e) (5 points) $y' - t^3 y + t^3 = 0$.

Solution: This is a first order linear equation. In standard form it is

$$y' - t^3 y = -t^3.$$

An integrating factor is therefore

$$\begin{aligned} \mu(t) &= e^{\int -s^3 \text{ d}s} \\ &= e^{\left(-\frac{t^4}{4}\right)}. \end{aligned}$$

The general solution is then

$$\begin{aligned} y(t) &= \frac{C + \int_0^t -t^3 e^{-\frac{s^4}{4}} \, ds}{e^{-\frac{t^4}{4}}}. \\ y(t) &= \boxed{C e^{\frac{t^4}{4}} + 1} \end{aligned}$$

(f) (5 points) $y'' - 4y' + 5y = t^2$.

Solution: This is a nonhomogeneous equation. We will solve it using the method of undetermined coefficients. The homogeneous equation

$$y'' - 4y' + 5y = 0$$

has characteristic equation

$$r^2 - 4r + 5 = 0$$

with roots

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

Thus the solution to the homogeneous equation is

$$C_1 e^{2t} \cos t + C_2 e^{2t} \sin t$$

First we will find a specific solution of the form $Y(t) = At^2 + Bt + C$.

$$\begin{aligned} Y(t) &= At^2 + Bt + C \\ Y'(t) &= 2At + B \\ Y''(t) &= 2A \end{aligned}$$

Thus the differential equation becomes

$$2A - 4(2At + B) + 5(At^2 + Bt + C) = t^2$$

which yields the equations

$$\begin{aligned} 5A &= 1 \\ -8A + 5B &= 0 \\ 2A - 4B + 5C &= 0 \end{aligned}$$

Hence $A = \frac{1}{5}$, $B = \frac{8}{25}$ and $C = \frac{22}{125}$.
Thus the general solution is

$$\boxed{y = \frac{1}{5}t^2 + \frac{8}{25}t + \frac{22}{125} + C_1 e^{2t} \cos t + C_2 e^{2t} \sin t.}$$

(g) (5 points) $y'' - 4y' + 4y = te^{2t}$.

Solution: This is a nonhomogeneous equation. We will solve it using the method of undetermined coefficients. The homogeneous equation

$$y'' - 4y' + 4y = 0$$

has characteristic equation

$$r^2 - 4r + 4 = (r - 2)^2 = 0$$

with the double root

$$r = 2.$$

Thus the solution to the homogeneous equation is

$$C_1 e^{2t} + C_2 t e^{2t}$$

First we will find a specific solution of the form $Y(t) = At^3 e^{2t} + Bt^2 e^{2t}$.

$$\begin{aligned} Y(t) &= At^3 e^{2t} + Bt^2 e^{2t} \\ Y'(t) &= 2At^3 e^{2t} + 3At^2 e^{2t} + 2Bt^2 e^{2t} + 2Bte^{2t} \\ &= 2At^3 e^{2t} + (3A + 2B)t^2 e^{2t} + 2Bte^{2t} \\ Y''(t) &= 4At^3 e^{2t} + 6At^2 e^{2t} + 2(3A + 2B)t^2 e^{2t} + 2(3A + 2B)te^{2t} + 4Bte^{2t} + 2Be^{2t} \\ &= 4At^3 e^{2t} + (12A + 4B)t^2 e^{2t} + (6A + 8B)te^{2t} + 2Be^{2t} \end{aligned}$$

Thus the differential equation becomes

$$4At^3e^{2t} + (12A+4B)t^2e^{2t} + (6A+8B)te^{2t} + 2Be^{2t}$$
$$-4\left(2At^3e^{2t} + (3A+2B)t^2e^{2t} + 2Bte^{2t}\right) + 4\left(At^3e^{2t} + Bt^2e^{2t}\right) = te^{2t}$$

which yields the equations

$$\begin{aligned} 4A - 8A + 4A &= 0 \\ (12A+4B) + (12A-8B) + 4B &= 0 \\ (6A+8B) - 8B &= 1 \\ 2B &= 0 \end{aligned}$$

Hence $A = \frac{1}{6}$, $B = 0$.
Thus the general solution is

$$\boxed{y = \tfrac{1}{6}t^3e^{2t} + C_1e^{2t} + C_2te^{2t}.}$$

(h) (5 points) $y'' - 5y' - 6y = 3t - e^{-t}$.

Solution: This is a nonhomogeneous equation. We will solve it using the method of undetermined coefficients. The homogeneous equation

$$y'' - 5y' - 6y = 0$$

has characteristic equation

$$r^2 - 5r - 6 = 0$$

with roots

$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm \sqrt{49}}{2} = -1 \text{ or } 6$$

Thus the solution to the homogeneous equation is

$$C_1e^{-t} + C_2e^{6t}$$

First we will find a specific solution Y_1 to the equation $y'' - 5y' - 6y = 3t$ of the form $Y_1(t) = At + B$.

$$\begin{aligned} Y_1(t) &= At + B \\ Y_1'(t) &= A \\ Y_1''(t) &= 0 \end{aligned}$$

Thus the differential equation becomes

$$0 - 5A - 6(At + B) = 3t$$

which yields the equations

$$\begin{aligned} -6A &= 3 \\ -5A + B &= 0 \end{aligned}$$

with solution $A = -\frac{1}{2}$ and $B = -\frac{5}{2}$ giving $Y_1(t) = -\frac{1}{2}t - \frac{5}{2}$.
Next we will find a specific solution Y_2 to the equation $y'' - 5y' - 6y = -e^{-t}$ of the form $Y_2(t) = Ate^{-t}$.

$$\begin{aligned} Y_2(t) &= Ate^{-t} \\ Y_2'(t) &= -Ate^{-t} + Ae^{-t} \\ Y_2''(t) &= Ate^{-t} - Ae^{-t} - Ae^{-t} = Ate^{-t} - 2Ae^{-t} \end{aligned}$$

Thus the differential equation becomes

$$Ate^{-t} - 2Ae^{-t} - 5(-Ate^{-t} + Ae^{-t}) - 6(Ate^{-t}) = -e^{-t}$$

which yields the equations

$$\begin{aligned} A + 5A - 6A &= 0 \\ -2A - 5A &= -1 \end{aligned}$$

with solution $A = \frac{1}{7}$ giving $Y_2(t) = \frac{1}{7}te^{-t}$.
A particular solution to the original equation is then $Y(t) = Y_1(t) + Y_2(t) = -\frac{1}{2}t - \frac{5}{2} + \frac{1}{7}te^{-t}$. The general solution is

$$\boxed{y(t) = -\frac{1}{2}t - \frac{5}{2} + \frac{1}{7}te^{-t} + C_1e^{-t} + C_2e^{6t}.}$$

(i) (5 points) $y'' - 4y' + 13y = e^t \cos t$.

Solution: This is a nonhomogeneous equation. We will solve it using the method of undetermined coefficients. The homogeneous equation

$$y'' - 4y' + 13y = 0$$

has characteristic equation

$$r^2 - 4r + 13 = 0$$

with roots

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

Thus the solution to the homogeneous equation is

$$C_1e^{2t}\cos 3t + C_2e^{2t}\sin 3t$$

First we will find a specific solution of the form $Y(t) = Ae^t \cos t + Be^t \sin t$.

$$\begin{aligned} Y(t) &= Ae^t \cos t + Be^t \sin t \\ Y'(t) &= Ae^t \cos t - Ae^t \sin t + Be^t \sin t + Be^t \cos t \\ &= (A + B)e^t \cos t + (-A + B)e^t \sin t \\ Y''(t) &= (A + B)e^t \cos t - (A + B)e^t \sin t + (-A + B)e^t \sin t + (-A + B)e^t \cos t \\ &= 2Be^t \cos t - 2Ae^t \sin t \end{aligned}$$

Thus the differential equation becomes

$$2Be^t\cos t - 2Ae^t\sin t - 4\Big((A+B)e^t\cos t + (-A+B)e^t\sin t\Big) + 13\Big(Ae^t\cos t + Be^t\sin t\Big) = e^t\cos t$$

which yields the equations

$$\begin{aligned} 2B - 4(A+B) + 13A &= 1 \\ -2A - 4(-A+B) + 13B &= 0 \end{aligned}$$

which simplify to the equations

$$\begin{aligned} -2B + 9A &= 1 \\ 2A + 9B &= 0 \end{aligned}$$

Hence $A = \frac{9}{85}$ and $B = -\frac{2}{85}$.
Thus the general solution is

$$\boxed{y = \tfrac{9}{85}e^t\cos t - \tfrac{2}{85}e^t\sin t + C_1e^{2t}\cos 3t + C_2e^{2t}\sin 3t.}$$

7. Find solutions for the following initial value problems

(a) (5 points) $y'' + 3y' = 0, \quad y(0) = 3, \quad y'(0) = 1.$

Solution: This is a second order linear homogeneous equation with constant coefficients. The characteristic equation is

$$\begin{aligned} r^2 + 3r &= 0 \\ (r+3)r &= 0. \end{aligned}$$

Thus $r_1 = -3$ and $r_2 = 0$ are roots.
The general solution is therefore

$$y(t) = C_1e^{-3t} + C_2e^{0t} = C_1e^{-3t} + C_2$$

and

$$y'(t) = -3c_1e^{-3t}.$$

The initial conditions give the equations

$$\begin{aligned} 3 = y(0) &= C_1e^{-3\cdot 0} + C_2 = C_1 + C_2 \\ 1 = y'(0) &= -3C_1e^{-3\cdot 0} + C_2 = -3C_1. \end{aligned}$$

Thus

$$\begin{aligned} C_1 &= -\tfrac{1}{3} \\ C_2 &= 3 - C_1 = \tfrac{10}{3} \end{aligned}$$

The solution to the IVP is therefore

$$\boxed{y(t) = -\tfrac{1}{3}e^{-3t} + \tfrac{10}{3}.}$$

(b) (5 points) $y'' + 6y' + 10y = 0, \quad y(0) = 1, \quad y'(0) = 0.$

Solution: This is a second order linear homogeneous equation with constant coefficients. The characteristic equation is

$$r^2 + 6r + 10 = 0$$

which has roots

$$r = \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i.$$

The general solution is therefore

$$\begin{aligned} y(t) &= C_1 e^{-3t}\cos t + C_2 e^{-3t}\sin t \\ &= e^{-3t}(C_1 \cos t + C_2 \sin t) \end{aligned}$$

and

$$\begin{aligned} y'(t) &= -3e^{-3t}(C_1\cos t + C_2 \sin t) + e^{-3t}(-C_1 \sin t + C_2 \cos t) \\ &= e^{-3t}((-3C_1 + C_2)\cos t + (-C_1 - 3C_2)\sin t). \end{aligned}$$

The initial conditions give the equations

$$\begin{aligned} 1 &= y(0) = C_1 e^{-3\cdot 0}\cos 0 + C_2 e^{-3\cdot 0}\sin 0 = C_1 \\ 0 &= y'(0) = e^{-3\cdot 0}((-3C_1 + C_2)\cos 0 + (-C_1 - 3C_2)\sin 0) = -3C_1 + C_2 \end{aligned}$$

Thus

$$\begin{aligned} C_1 &= 1 \\ C_2 &= 3C_1 = 3 \end{aligned}$$

The solution to the IVP is therefore

$$\boxed{y(t) = e^{-3t}\cos t + 3e^{-3t}\sin t.}$$

(c) (5 points) $ty' + y = 6, \quad y(1) = 3.$

Solution: This is a first order linear equation. In standard form it is

$$y' + \frac{1}{t}y = \frac{6}{t}.$$

For $t > 0$ An integrating factor is therefore

$$\begin{aligned} \mu(t) &= e^{\int_1^t \frac{1}{s}\, \mathrm{d}s} \\ &= e^{\ln t} \\ &= t \end{aligned}$$

The solution of the IVP is therefore

$$\begin{aligned} y(t) &= \frac{3 + \int_1^t s \cdot \frac{6}{s}\, ds}{t} \\ &= \frac{3 + \int_1^t 6\, ds}{t} \\ &= \boxed{\frac{6t-3}{t}.} \end{aligned}$$

(d) (5 points) $y'' + 8y' + 20y = 0, \quad y(0) = 6, \quad y'(0) = -1.$

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$r^2 + 8r + 20 = 0$$

which has roots

$$r = \frac{-8 \pm \sqrt{8^2 - 4(1)(20)}}{2(1)} = \frac{-8 \pm \sqrt{-16}}{2} = -4 \pm 2i.$$

The general solution is therefore

$$\begin{aligned} y(t) &= C_1 e^{-4t} \cos 2t + C_2 e^{-4t} \sin 2t \\ &= e^{-4t}(C_1 \cos 2t + C_2 \sin 2t) \end{aligned}$$

and

$$\begin{aligned} y'(t) &= -4e^{-4t}(C_1 \cos 2t + C_2 \sin 2t) + e^{-4t}(-2C_1 \sin 2t + 2C_2 \cos 2t) \\ &= e^{-4t}((-4C_1 + 2c_2)\cos 2t + (-2C_1 - 4C_2)\sin 2t). \end{aligned}$$

The initial conditions give the equations

$$\begin{aligned} 6 &= y(0) = C_1 e^{-4 \cdot 0} \cos 2 \cdot 0 + C_2 e^{-4 \cdot 0} \sin 2 \cdot 0 = C_1 \\ -1 &= y'(0) = e^{-4 \cdot 0}((-4C_1 + 2C_2)\cos 2 \cdot 0 + (-2C_1 - 4C_2)\sin 2 \cdot 0) = -4C_1 + 2C_2 \end{aligned}$$

Thus

$$\begin{aligned} C_1 &= 6 \\ C_2 &= \frac{-1 + 4C_1}{2} = \frac{-1 + 4 \cdot 6}{2} = \tfrac{23}{2} \end{aligned}$$

The solution to the IVP is therefore

$$\boxed{y(t) = 6e^{-4t}\cos 2t + \tfrac{23}{2}e^{-4t}\sin 2t.}$$

(e) (5 points) $y'' + 3y' = e^{-3t}, \quad y(0) = 0, \quad y'(0) = 1.$

Solution: This is a second order linear nonhomogeneous equation with constant coefficients. The characteristic equation for the homogeneous part is

$$r^2 + 3r = 0$$
$$(r+3)r = 0.$$

Thus $r_1 = -3$ and $r_2 = 0$ are roots.
The homogeneous solution is therefore

$$C_1e^{-3t} + C_2e^{0t} = C_1e^{-3t} + C_2$$

First we will find a specific solution of the form $Y(t) = Ate^{-3t}$.

$$\begin{aligned} Y(t) &= Ate^{-3t} \\ Y'(t) &= -3Ate^{-3t} + Ae^{-3t} \\ Y''(t) &= 9Ate^{-3t} - 3Ae^{-3t} - 3Ae^{-3t} \\ &= 9Ate^{-3t} - 6Ae^{-3t} \end{aligned}$$

Thus the differential equation becomes

$$9Ate^{-3t} - 6Ae^{-3t} + 3\Big(-3Ate^{-3t} + Ae^{-3t}\Big) = e^{-3t}$$

which yields the equations

$$9A - 9A = 0$$
$$-6A + 3A = 1$$

Hence $A = -\frac{1}{3}$.
Thus the general solution is

$$y = -\tfrac{1}{3}te^{-3t} + C_1e^{-3t} + C_2$$

with derivative

$$y' = te^{-3t} - \tfrac{1}{3}e^{-3t} - 3C_1e^{-3t}$$

The initial conditions give the equations

$$0 = y(0) = -\tfrac{1}{3}\cdot 0 \cdot e^{-3\cdot 0} + C_1e^{-3\cdot 0} + C_2 = C_1 + C_2$$
$$1 = y'(0) = 0 \cdot e^{-3\cdot 0} - \tfrac{1}{3}e^{-3\cdot 0} - 3C_1e^{-3\cdot 0} = -\tfrac{1}{3} - 3C_1$$

Thus

$$C_1 = -\tfrac{4}{9}$$
$$C_2 = \tfrac{4}{9}$$

The solution to the IVP is therefore

$$\boxed{y(t) = -\tfrac{1}{3}te^{-3t} - \tfrac{4}{9}e^{-3t} + \tfrac{4}{9}.}$$

(f) (5 points) $y'' - y' - 5y = 0, \quad y(0) = e, \quad y'(0) = 0.$

Solution: This is a second order linear homogeneous equation with constant coefficients. The characteristic equation is

$$r^2 - r - 5 = 0$$

which has roots

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-5)}}{2(1)} = \frac{1 \pm \sqrt{21}}{2}.$$

The general solution is therefore

$$y(t) = C_1 e^{\frac{1+\sqrt{21}}{2} \cdot t} + C_2 e^{\frac{1-\sqrt{21}}{2} \cdot t}$$

and

$$y'(t) = \frac{1+\sqrt{21}}{2} \cdot C_1 e^{\frac{1+\sqrt{21}}{2} \cdot t} + \frac{1-\sqrt{21}}{2} \cdot C_2 e^{\frac{1-\sqrt{21}}{2} \cdot t}$$

The initial conditions give the equations

$$e = y(0) = C_1 e^{\frac{1+\sqrt{21}}{2} \cdot 0} + C_2 e^{\frac{1-\sqrt{21}}{2} \cdot 0} = C_1 + C_2$$
$$0 = y'(0) = \frac{1+\sqrt{21}}{2} \cdot C_1 e^{\frac{1+\sqrt{21}}{2} \cdot 0} + \frac{1-\sqrt{21}}{2} \cdot C_2 e^{\frac{1-\sqrt{21}}{2} \cdot 0} = \frac{1+\sqrt{21}}{2} \cdot C_1 + \frac{1-\sqrt{21}}{2} \cdot C_2$$

Thus

$$C_1 = \frac{e(\sqrt{21}-1)}{2\sqrt{21}}$$
$$C_2 = e - \frac{e(\sqrt{21}-1)}{2\sqrt{21}}$$

The solution to the IVP is therefore

$$\boxed{y(t) = \frac{e(\sqrt{21}-1)}{2\sqrt{21}} \cdot e^{\frac{1+\sqrt{21}}{2} \cdot t} + \left(e - \frac{e(\sqrt{21}-1)}{2\sqrt{21}}\right) e^{\frac{1-\sqrt{21}}{2} \cdot t}.}$$

You can expect nicer constants on the midterm!

(g) (5 points) $y'' + 7y = 6, \quad y(0) = 0, \quad y'(0) = 3.$

Solution: This is a second order linear nonhomogeneous equation with constant coefficients. The characteristic equation for the homogeneous part is

$$r^2 + 7 = 0$$
$$(r+3)r = 0.$$

Thus $r = \pm\sqrt{7}i$ are roots.
The homogeneous solution is therefore

$$C_1 \cos\sqrt{7}t + C_2 \sin\sqrt{7}t$$

First we will find a specific solution of the form $Y(t) = A$.

$$Y(t) = A$$
$$Y'(t) = 0$$
$$Y''(t) = 0$$

Thus the differential equation becomes

$$0 + 7A = 6$$

Hence $A = \frac{6}{7}$.
Thus the general solution is

$$y = \frac{6}{7} + C_1 \cos\sqrt{7}t + C_2 \sin\sqrt{7}t$$

with derivative

$$y' = -\sqrt{7}C_1 \sin\sqrt{7}t + \sqrt{7}C_2 \cos\sqrt{7}t$$

The initial conditions give the equations

$$0 = y(0) = \frac{6}{7} + C_1 \cos\sqrt{7}\cdot 0 + C_2 \sin\sqrt{7}\cdot 0 = \frac{6}{7} + C_1$$
$$3 = y'(0) = -\sqrt{7}C_1 \sin\sqrt{7}\cdot 0 + \sqrt{7}C_2 \cos\sqrt{7}\cdot 0 = \sqrt{7}C_2$$

Thus

$$C_1 = -\frac{6}{7}$$
$$C_2 = \frac{3}{\sqrt{7}}$$

The solution to the IVP is therefore

$$\boxed{y(t) = \frac{6}{7} - \frac{6}{7}\cos\sqrt{7}t + \frac{3}{\sqrt{7}}\sin\sqrt{7}t.}$$

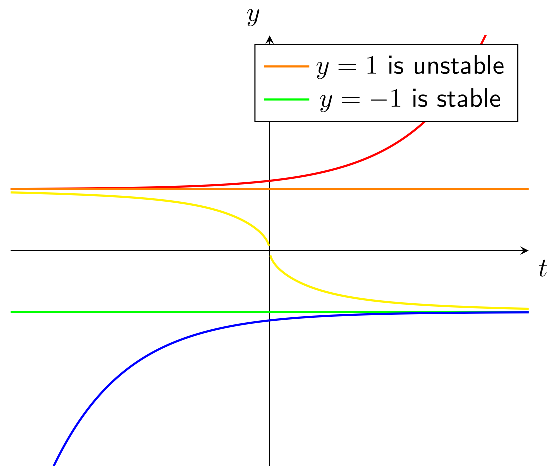
8. (10 points) Sketch solutions to the equation

$$y' = \ln|y|.$$

on the (y, t)-plane. Be sure to include all equilibrium solutions and note whether they are stable, unstable or semistable. Also include a nonequilibrium solution above and below each equilibrium solution.

Solution: This is an autonomous equation. The equilibrium solutions will occur at roots of the equation

$$\begin{aligned}0 &= \ln|y| \\ e^0 &= e^{\ln|y|} \\ 1 &= |y| \\ y &= \pm 1\end{aligned}$$



9. (10 points) t^2 is a solution to the differential equation

$$3t^2y'' - 2ty' - 2y = 0.$$

Find the general solution.

Solution: Since we're given a solution t^2 to the differential equation we can look for a solution of the form $t^2u(t)$. Suppose $t^2u(t)$ satisfies the differential equation above.

$$\begin{aligned}\frac{d}{dt}(t^2u(t)) &= 2tu(t) + t^2u'(t) \\ \frac{d^2}{dt^2}(t^2u(t)) &= 2u(t) + 4tu'(t) + t^2u''(t)\end{aligned}$$

thus

$$\begin{aligned}3t^2(2u(t) + 4tu'(t) + t^2u''(t)) - 2t(2tu(t) + t^2u'(t)) - 2t^2u(t) &= 0 \\ 6t^2u(t) + 12t^3u'(t) + 3t^4u''(t) - 4t^2u(t) - 2t^3u'(t) - 2t^2u(t) &= 0 \\ 3t^4u''(t) + (12t^3 - 2t^3)u'(t) + (6t^2 - 4t^2 - 2t^2)u(t) &= 0 \\ 3t^4u''(t) + 10t^3u'(t) &= 0 \\ u''(t) + \frac{10}{3t}u'(t) &= 0\end{aligned}$$

This is a first order linear equation for the function u' with integrating factor

$$\mu(t) = e^{\int_1^t \frac{10}{3s}\,ds} = e^{\frac{10}{3}\ln t} = t^{10/3}$$

Thus

$$u'(t) = \frac{C + \int_1^t s^{10/3} \cdot 0\,ds}{t^{10/3}} = Ct^{-10/3}$$

$$u(t) = \int u'(t)\,dt = \int Ct^{-10/3}\,dt = C_1 t^{-7/3} + C_2$$

Therefore the general solution is

$$y(t) = (C_1 t^{-7/3} + C_2)t^2 = \boxed{C_1 t^{-1/3} + C_2 t^2}$$

10. (10 points) Suppose that the force of drag on a ball of mass m falling through the air were proportional to the velocity v of the ball. Set up a model for the velocity of the ball falling through the air if the acceleration of gravity is g. Find the velocity as a function of time if the initial velocity is 0.

Solution: At any given time we have two forces acting on the ball. The downward force of gravity with magnitude F_g and the upward force of drag with magnitude F_d. The force of gravity will be considered to be constant

$$F_g = mg.$$

The force of drag depends on the downward velocity v of the ball

$$F_d = kv.$$

The total downward force on the ball is therefore $F = F_g - F_d$. Newton's Second Law asserts that $F = ma$. Thus the acceleration $a = \frac{dv}{dt}$ of the ball is

$$\begin{aligned} \frac{dv}{dt} &= \frac{F}{m} \\ &= \frac{F_g - F_d}{m} \\ &= \frac{mg - kv}{m} \\ &= g - \frac{kv}{m} \end{aligned}$$

Our model for the velocity of the ball is therefore

$$\boxed{\frac{dv}{dt} = g - \frac{k}{m} \cdot v(t), \quad v(0) = 0}$$

where $v(t)$ is the velocity of the ball at time t, g is the acceleration of gravity, m is the mass of the ball, and k is a constant that depends on the viscosity of the atmosphere and size of the ball.

This is a first order linear equation with integrating factor

$$\mu(t) = e^{\int_0^t \frac{k}{m}\,\mathrm{d}s}$$

$$= e^{kt/m}$$

and solution

$$\begin{aligned} v(t) &= \frac{C + \int_0^t e^{ks/m} g\, ds}{e^{kt/m}} \\ &= Ce^{-kt/m} + \frac{mg}{k} \end{aligned}$$

In order to satisfy the intial condition $v(0) = 0$ we solve for C with $t = 0$ to get

$$\begin{aligned} 0 &= Ce^0 + \frac{mg}{k} \\ C &= -\frac{mg}{k} \end{aligned}$$

Thus the velocity at time t is

$$\boxed{v(t) = -\frac{mg}{k} e^{-kt/m} + \frac{mg}{k}}$$

11. (10 points) Show that the functions $y_1(t) = t$ and $y_2(t) = \sin(t)$ cannot both be solutions to a single differential equation of the form

$$y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on $\mathbf{R}$.

Solution: The Wronskian of y_1 and y_2 is

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix} \\ &= t\cos t - \sin t \end{aligned}$$

The Wronskian is either always 0 or always nonzero for two solutions of the same homogeneous second order linear equation. Notice that $W(y_1, y_2)(0) = 0\cos 0 - \sin 0 = 0$ but $W(y_1, y_2)(\pi) = \pi\cos\pi - \sin\pi = -\pi$. Thus y_1 and y_2 cannot be solutions to the same equation of the the form above.

12. (10 points) Use the method of successive approximations to compute the approximation $\phi_3(t)$ to the initial value problem

$$y' = 3ty - t^2, \qquad y(0) = 0$$

starting with the initial approximation $\phi_0(t) = 0$.

Solution:

$$\begin{aligned}
\phi_1(t) &= \int_0^t 3s \cdot 0 - s^2 \, ds \\
&= \int_0^t -s^2 \, ds \\
&= \frac{-s^3}{3} \Big|_{s=0}^{t} \\
&= \frac{-t^3}{3}
\end{aligned}$$

$$\begin{aligned}
\phi_2(t) &= \int_0^t 3s \cdot \left(\frac{-s^3}{3}\right) - s^2 \, ds \\
&= \int_0^t -s^4 - s^2 \, ds \\
&= \frac{-s^5}{5} - \frac{s^3}{3} \Big|_{s=0}^{t} \\
&= \frac{-t^5}{5} - \frac{t^3}{3}
\end{aligned}$$

$$\begin{aligned}
\phi_3(t) &= \int_0^t 3s \cdot \left(\frac{-s^5}{5} - \frac{s^3}{3}\right) - s^2 \, ds \\
&= \int_0^t -\frac{3}{5}s^6 - s^4 - s^2 \, ds \\
&= \frac{-3s^7}{7 \cdot 5} - \frac{s^5}{5} - \frac{s^3}{3} \Big|_{s=0}^{t} \\
&= \boxed{\tfrac{-3t^7}{35} - \tfrac{t^5}{5} - \tfrac{t^3}{3}}
\end{aligned}$$

13. (10 points) Give a closed form expression for the solution to the first order difference equation

$$y_{n+1} = 2y_n + n, \qquad y_0 = 1$$

Solution:

$$\begin{aligned}
y_0 &= 1 \\
y_1 &= 2y_0 + 0 = 2 \cdot 1 \\
y_2 &= 2y_1 + 1 = 2 \cdot 2 \cdot 1 + 1 \\
y_3 &= 2y_2 + 2 = 2 \cdot 2 \cdot 2 \cdot 1 + 2 \cdot 1 + 2 \\
y_4 &= 2y_3 + 3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 + 2 \cdot 2 \cdot 1 + 2 \cdot 2 + 3 \\
y_5 &= 2y_4 + 4 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 1 + 2 \cdot 2 \cdot 2 \cdot 1 + 2 \cdot 2 \cdot 2 + 2 \cdot 3 + 4
\end{aligned}$$

$$\vdots$$

$$\begin{aligned} y_n &= 2^n + \sum_{k=1}^{n-1} k 2^{n-k-1} \\ &= 2^n + \sum_{k=0}^{n-1} k 2^{n-1-k} \\ &= 2^n + \sum_{k=0}^{n-1} (n-1-k) 2^k \\ &= \boxed{2^n + \tfrac{n-1}{1-2} - \tfrac{2-2^n}{(1-2)^2}} \end{aligned}$$

14. (10 points) Find the general solution to

$$t^2 y'' - 2y = 3t^2$$

for $t > 0$ given that the general solution to the homogeneous equation $t^2 y'' - 2y = 0$ for $t > 0$ is

$$C_1 t^2 + C_2 t^{-1}.$$

Solution: In standard format the ODE is

$$y'' + 0y' - \tfrac{2}{t^2} y = 3$$

Let $y_1(t) = t^2$ and $y_2(t) = t^{-1}$. The Wronskian is then

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} \\ &= -1 - 2 \\ &= -3 \end{aligned}$$

Using variation of parameters we get a solution

$$\begin{aligned} Y(t) &= -y_1(t) \int_{t_0}^{t} \tfrac{y_2(s) g(s)}{W(y_1,y_2)(s)}\, ds + y_2(t) \int_{t_0}^{t} \tfrac{y_1(s) g(s)}{W(y_1,y_2)(s)}\, ds \\ &= -t^2 \int_1^t \tfrac{s^{-1} \cdot 3}{-3}\, ds + t^{-1} \int_1^t \tfrac{s^2 \cdot 3}{-3}\, ds \\ &= -t^2 \int_1^t -s^{-1}\, ds + t^{-1} \int_1^t -s^2\, ds \end{aligned}$$

$$
\begin{aligned}
&= -t^2(-\ln|s|)\Big|_{s=1}^{t} + t^{-1}(-\tfrac{s^3}{3})\Big|_{s=1}^{t} \\
&= -t^2(-\ln t + \ln 1) + t^{-1}(-\tfrac{t^3}{3} + \tfrac{1}{3}) \\
&= t^2 \ln t - \tfrac{t^2}{3} + \tfrac{t^{-1}}{3}
\end{aligned}
$$

and general solution is

$$
\begin{aligned}
Y(t) + C_1 y_1(t) + C_2 y_2(t) &= t^2 \ln t - \tfrac{t^2}{3} + \tfrac{t^{-1}}{3} + C_1 t^2 + C_2 t^{-1} \\
&= \boxed{t^2 \ln t + K_1 t^2 + K_2 t^{-1}}
\end{aligned}
$$