Practice Midterm 2 Solutions – Math 2255

** **Practice midterm I** provides good practice problems for previous material.
** **Bring a single double-sided** 8.5×11 **sheet of notes to use during the final**.

1. Decide if the following statements are TRUE or FALSE. **You do NOT need to justify your answers.**
   (a) (2 points) If $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$ then $\mathcal{L}\{f(t)g(t)\} = F(s)G(s)$

Solution: **F** ($\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$.)

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**
   (a) (2 points) Give an example of a piecewise continuous function on the interval $[0, 5]$ which is not continuous on the interval $[0, 5]$.

Solution: The Heavyside function $u_2(t)$.

3. (10 points) Using **only the definition of the Laplace transform** compute the Laplace transform $\mathcal{L}\{u_2(t)t\}$.

Solution:

$$\begin{aligned}
\mathcal{L}\{u_2(t)t\} &= \int_0^\infty e^{-st}u_2(t)t\,\mathrm{d}t \\
&= \lim_{A\to\infty}\int_0^A e^{-st}u_2(t)t\,\mathrm{d}t \\
&= \lim_{A\to\infty}\int_2^A te^{-st}\,\mathrm{d}t \\
&\quad u = t, \quad \mathrm{d}v = e^{-st}\,\mathrm{d}t, \quad \mathrm{d}u = \mathrm{d}t, \quad v = \tfrac{e^{-st}}{-s} \\
&= \lim_{A\to\infty}\left(\left.\frac{te^{-st}}{-s}\right|_{t=2}^{A} - \int_2^A \frac{e^{-st}}{-s}\,\mathrm{d}t \right) \\
&= \lim_{A\to\infty}\left(\frac{Ae^{-sA}}{-s} - \frac{2e^{-2s}}{-s} - \left.\frac{e^{-st}}{s^2}\right|_{t=2}^{A} \right) \\
&= \lim_{A\to\infty}\left(\frac{Ae^{-sA}}{-s} - \frac{2e^{-2s}}{-s} - \frac{e^{-sA}}{s^2} + \frac{e^{-2s}}{s^2} \right) \\
&= \lim_{A\to\infty}\left(\frac{Ae^{-sA}}{-s} \right) + \frac{2e^{-2s}}{s} - \lim_{A\to\infty}\left(\frac{e^{-sA}}{s^2} \right) + \frac{e^{-2s}}{s^2} \\
&= \lim_{A\to\infty}\left(\frac{A}{-se^{sA}} \right) + \frac{2e^{-2s}}{s} - 0 + \frac{e^{-2s}}{s^2}, \qquad s > 0 \\
&\quad \text{L'Hôpital's Rule} \\
&= \lim_{A\to\infty}\left(\frac{1}{-s^2e^{sA}} \right) + \frac{2e^{-2s}}{s} - 0 + \frac{e^{-2s}}{s^2}, \qquad s > 0 \\
&= 0 + \frac{2e^{-2s}}{s} - 0 + \frac{e^{-2s}}{s^2}, \qquad s > 0 \\
&= \boxed{\left(\tfrac{2}{s} + \tfrac{1}{s^2}\right)e^{-2s}, \quad s > 0}
\end{aligned}$$

4. (10 points) Find the inverse Laplace transform $\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)} + \frac{1}{(s-4)^3}\right\}$.

Solution: Using partial fractions

$$\begin{aligned}\frac{1}{(s-2)(s-3)} &= \frac{A}{s-2} + \frac{B}{s-3} \\ 1 &= A(s-3) + B(s-2) \\ 1 &= As - 3A + Bs - 2B\end{aligned}$$

Constant term gives the equation

$$1 = -3A - 2B.$$

Coefficient of s gives equation

$$0 = A + B.$$

Thus $B = -A$ and hence $1 = -3A + 2A$ giving $A = -1$ and $B = 1$. Therefore

$$\frac{1}{(s-2)(s-3)} = \frac{-1}{s-2} + \frac{1}{s-3}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s-3)} + \frac{1}{(s-4)^3}\right\} &= \mathcal{L}^{-1}\left\{\frac{-1}{s-2} + \frac{1}{s-3} + \frac{1}{(s-4)^3}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-1}{s-2} + \frac{1}{s-3} + \frac{1}{2} \cdot \frac{2!}{(s-4)^{2+1}}\right\} \\ &= \boxed{-e^{2t} + e^{3t} + \frac{1}{2}t^2 e^{4t}}\end{aligned}$$

5. (10 points) Express the function

$$f(t) = \begin{cases} 3, & 0 \le t < 2 \\ 6, & 2 \le t < 3 \\ -5, & 3 \le t \end{cases}$$

as a linear combination of Heavyside functions $u_c(t)$.

Solution:

$$\begin{aligned}f(t) &= 3(u_0(t) - u_2(t)) + 6(u_2(t) - u_3(t)) - 5u_3(t) \\ &= 3u_0(t) - 3u_2(t) + 6u_2(t) - 6u_3(t) - 5u_3(t) \\ &= \boxed{3u_0(t) + 3u_2(t) - 11u_3(t)}\end{aligned}$$

6. (10 points) Express the function

$$g(t) = \begin{cases} \sin t, & 0 \le t < 4 \\ t^2, & 4 \le t < 7 \\ 8t, & 7 \le t \end{cases}$$

as sum of products of Heavyside functions $u_c(t)$ with continuous functions.

Solution:

$$\begin{aligned} f(t) &= \sin t(u_0(t) - u_4(t)) + t^2(u_4(t) - u_7(t)) + 8tu_7(t) \\ &= (\sin t)u_0(t) - (\sin t)u_4(t) + t^2 u_4(t) - t^2 u_7(t) + 8tu_7(t) \\ &= \boxed{(\sin t)u_0(t) + (t^2 - \sin t)u_4(t) + (8t - t^2)u_7(t)} \end{aligned}$$

7. Find the general solution to

$$t^2 y'' - 2y = 3t^2 - 1$$

for $t > 0$ given that the general solution to the homogeneous equation $t^2y'' - 2y = 0$ for $t > 0$ is

$$c_1 t^2 + c_2 t^{-1}.$$

Solution: This is nonhomogeneous equation with nonconstant coefficients for which we have the homogeneous solution. Therefore we can use variation of parameters to find a particular solution. Set

$$y_1(t) = t^{-1}, \qquad y_2(t) = t^2$$

The Wronskian is then

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} t^{-1} & t^2 \\ -t^{-2} & 2t \end{vmatrix} \\ &= t^{-1}(2t) - (-t^{-2})t^2 \\ &= 3 \end{aligned}$$

Before we can use variation of parameters we must rewrite the linear equation in the standard form:

$$y'' - \tfrac{2}{t^2}y = 3 - \tfrac{1}{t^2}.$$

So we see that $g(t) = 3 - t^{-2}$. Using the variation of parameters formula we then get a particular solution

$$\begin{aligned} Y(t) &= -y_1(t)\int_1^t \frac{y_2(s)g(s)}{W(y_1,y_2)(s)}\,\mathrm{d}s + y_2(t)\int_1^t \frac{y_1(s)g(s)}{W(y_1,y_2)(s)}\,\mathrm{d}s \\ &= -t^{-1}\int_1^t \tfrac{1}{3}s^2(3 - s^{-2})\,\mathrm{d}s + t^2\int_1^t \tfrac{1}{3}s^{-1}(3 - s^{-2})\,\mathrm{d}s \\ &= -\tfrac{1}{3}t^{-1}\left(\int_1^t 3s^2 - 1\,\mathrm{d}s\right) + \tfrac{1}{3}t^2\left(\int_1^t 3s^{-1} - s^{-3}\,\mathrm{d}s\right) \\ &= -\tfrac{1}{3}t^{-1}\left(s^3 - s\right)\Big|_{s=1}^t + \tfrac{1}{3}t^2\left(3\ln s + \tfrac{1}{2}s^{-2}\right)\Big|_{s=1}^t \\ &= -\tfrac{1}{3}t^{-1}\left(t^3 - t - 1^3 + 1\right) + \tfrac{1}{3}t^2\left(3\ln t + \tfrac{1}{2}t^{-2} - 3\ln 1 - \tfrac{1}{2}\cdot 1^{-2}\right) \\ &= -\tfrac{1}{3}t^2 + \tfrac{1}{3} + t^2\ln t + \tfrac{1}{6} - \tfrac{1}{6}t^2 \\ &= -\tfrac{1}{2}t^2 + \tfrac{1}{2} + t^2\ln t \end{aligned}$$

The general solution is therefore

$$\begin{aligned} y(t) &= Y(t) + C_1y_1(t) + C_2y_2(t) \\ &= -\tfrac{1}{2}t^2 + \tfrac{1}{2} + t^2\ln t + C_1t^{-1} + C_2t^2 \\ &= \boxed{\tfrac{1}{2} + t^2\ln t + K_1t^{-1} + K_2t^2} \end{aligned}$$

8. (10 points) Compute the convolution $f * g$ if

$$f(t) = \begin{cases} 0, & t \leq -1 \\ 1 - t^2, & -1 \leq t \leq 1 \\ 0, & 1 \leq t \end{cases}$$

and $g(t) = t$.

Solution:

$$\begin{aligned}
(f * g)(t) &= \int_0^t f(x)g(t-x)\,\mathrm{d}x \\
&= \begin{cases} \int_0^t f(x)(t-x)\,\mathrm{d}x, & t \leq -1 \\ \int_0^t f(x)(t-x)\,\mathrm{d}x, & -1 \leq t \leq 1 \\ \int_0^t f(x)(t-x)\,\mathrm{d}x, & 1 \leq t \end{cases} \\
&= \begin{cases} \int_0^{-1}(1-x^2)(t-x)\,\mathrm{d}x + \int_{-1}^t 0\cdot(t-x)\,\mathrm{d}x, & t \leq -1 \\ \int_0^t(1-x^2)(t-x)\,\mathrm{d}x, & -1 \leq t \leq 1 \\ \int_0^1(1-x^2)(t-x)\,\mathrm{d}x + \int_1^t 0\cdot(t-x)\,\mathrm{d}x, & 1 \leq t \end{cases} \\
&= \begin{cases} \int_0^{-1}(1-x^2)(t-x)\,\mathrm{d}x, & t \leq -1 \\ \int_0^t(1-x^2)(t-x)\,\mathrm{d}x, & -1 \leq t \leq 1 \\ \int_0^1(1-x^2)(t-x)\,\mathrm{d}x, & 1 \leq t \end{cases} \\
&= \begin{cases} \int_0^{-1} t - x - tx^2 + x^3\,\mathrm{d}x, & t \leq -1 \\ \int_0^t t - x - tx^2 + x^3\,\mathrm{d}x, & -1 \leq t \leq 1 \\ \int_0^1 t - x - tx^2 + x^3\,\mathrm{d}x, & 1 \leq t \end{cases} \\
&= \begin{cases} tx - \frac{1}{2}x^2 - \frac{1}{3}tx^3 + \frac{1}{4}x^4\Big|_{x=0}^{-1}, & t \leq -1 \\ tx - \frac{1}{2}x^2 - \frac{1}{3}tx^3 + \frac{1}{4}x^4\Big|_{x=0}^{t}, & -1 \leq t \leq 1 \\ tx - \frac{1}{2}x^2 - \frac{1}{3}tx^3 + \frac{1}{4}x^4\Big|_{x=0}^{1}, & 1 \leq t \end{cases} \\
&= \begin{cases} t(-1) - \frac{1}{2}(-1)^2 - \frac{1}{3}t(-1)^3 + \frac{1}{4}(-1)^4, & t \leq -1 \\ t\cdot t - \frac{1}{2}t^2 - \frac{1}{3}t\cdot t^3 + \frac{1}{4}t^4, & -1 \leq t \leq 1 \\ t - \frac{1}{2} - \frac{1}{3}t + \frac{1}{4}, & 1 \leq t \end{cases} \\
&= \begin{cases} -t - \frac{1}{2} + \frac{1}{3}t + \frac{1}{4}, & t \leq -1 \\ t^2 - \frac{1}{2}t^2 - \frac{1}{3}t^4 + \frac{1}{4}t^4, & -1 \leq t \leq 1 \\ \frac{2}{3}t - \frac{1}{2}, & 1 \leq t \end{cases} \\
&= \boxed{\begin{cases} -\frac{2}{3}t - \frac{1}{4}, & t \leq -1 \\ \frac{1}{2}t^2 - \frac{1}{12}t^4, & -1 \leq t \leq 1 \\ \frac{2}{3}t - \frac{1}{4}, & 1 \leq t \end{cases}}
\end{aligned}$$

9. (10 points) Find the general solution to

$$y'' + 3y' - 2y = 6e^{2t}.$$

Solution: The homogeneous equation $y'' + 3y' - 2y = 0$ has characteristic polynomial

$$Z(r) = r^2 + 3r - 2$$

which has roots $\frac{-3 \pm \sqrt{17}}{2}$. Thus a fundamental set of solutions to the homogeneous equation is

$$y_1(t) = e^{\frac{-3-\sqrt{17}}{2}t}, \qquad y_2(t) = e^{\frac{-3+\sqrt{17}}{2}t}$$

We will use undetermined coefficients to find a particular solution $Y(t)$ to the nonhomogeneous equation of the form

$$Y(t) = Ae^{2t}.$$

Taking derivatives we get

$$Y'(t) = 2Ae^{2t}$$
$$Y''(t) = 4Ae^{2t}$$

so A must satisfy

$$4Ae^{2t} + 3 \cdot 2Ae^{2t} - 2Ae^{2t} = 6e^{2t}.$$

Therefore $8A = 6$ giving $A = \frac{3}{4}$ and $Y(t) = \frac{3}{4}e^{2t}$. The general solution is therefore

$$\begin{aligned} y(t) &= Y(t) + C_1 y_1(t) + C_2 y_2(t) \\ &= \boxed{\tfrac{3}{4}e^{2t} + C_1 e^{\frac{-3-\sqrt{17}}{2}t} + C_2 e^{\frac{-3+\sqrt{17}}{2}t}} \end{aligned}$$

10. (10 points) Find the general solution to

$$y^{(6)} + 9y^{(4)} + 27y'' + 27y = 0.$$

Solution: The homogeneous equation has characteristic polynomial

$$Z(r) = r^6 + 9r^4 + 27r^2 + 27 = (r^2 + 3)^3 = (r + i\sqrt{3})^3 (r - i\sqrt{3})^3$$

which has complex conjugate roots $\pm i\sqrt{3}$ each with multiplicity 3. Thus a general solution is therefore

$$y(t) = \boxed{C_1 \cos\sqrt{3}t + C_2 t\cos\sqrt{3}t + C_3 t^2 \cos\sqrt{3}t + C_4 \sin\sqrt{3}t + C_5 t\sin\sqrt{3}t + C_6 t^2 \sin\sqrt{3}t}$$

11. (10 points) Solve the initial value problem

$$y'' + 4y = g(t)$$

where

$$g(t) = \begin{cases} 0, & 0 \le t < 2 \\ 1, & 2 \le t \end{cases}$$

and

$$y(0) = 0, \qquad y'(0) = 0.$$

Solution: The function g is just the Heavyside function $g(t) = u_2(t)$ so we want a solution to the IVP

$$y'' + 4y = u_2(t), \qquad y(0) = 0, \qquad y(0) = 0.$$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the differential equation we get

$$\begin{aligned}
\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y(t)\} &= \mathcal{L}\{u_2(t)\} \\
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \frac{e^{-2s}}{s} \\
s^2Y(s) - s \cdot 0 - 0 + 4Y(s) &= \frac{e^{-2s}}{s} \\
(s^2+4)Y(s) &= \frac{e^{-2s}}{s} \\
Y(s) &= \frac{e^{-2s}}{s(s^2+4)}
\end{aligned}$$

We will need the inverse Laplace transform of $\frac{1}{s(s^2+4)}$ so first we use partial fractions:

$$\begin{aligned}
\frac{1}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} \\
1 &= A(s^2+4) + Bs^2 + Cs \\
1 &= As^2 + 4A + Bs^2 + Cs \\
1 &= (A+B)s^2 + Cs + 4A
\end{aligned}$$

Giving the three equations

$$\begin{aligned}
0 &= A + B \\
0 &= C \\
1 &= 4A
\end{aligned}$$

Thus $A = \frac{1}{4}$, $B = -\frac{1}{4}$ and $C = 0$. Therefore $\frac{1}{s(s^2+4)} = \frac{1/4}{s} + \frac{-s/4}{s^2+4}$.

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} &= \tfrac{1}{4}\mathcal{L}^{-1}\left\{\frac{1/4}{s} - \frac{s/4}{s^2+4}\right\} \\
&= \tfrac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \tfrac{1}{4}\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} \\
&= \tfrac{1}{4} - \tfrac{1}{4}\cos 2t.
\end{aligned}$$

Applying rule 13 from Table 6.2.1 we can conclude that

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s^2+4)}\right\} \\
&= \boxed{u_2(t)\left(\tfrac{1}{4} - \tfrac{1}{4}\cos 2(t-2)\right).}
\end{aligned}$$

12. (10 points) Solve the initial value problem

$$y'' + 4y' + 4y = \delta(t-3)$$

where δ is the Dirac delta function and

$$y(0) = 1, \qquad y'(0) = 0.$$

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$. Taking the Laplace transform of the differential equation we get

$$\begin{aligned}
\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 4\mathcal{L}\{y(t)\} &= \mathcal{L}\{\delta(t-3)\} \\
s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) &= e^{-3s} \\
s^2Y(s) - s\cdot 1 - 0 + 4sY(s) - 4\cdot 1 + 4Y(s) &= e^{-3s} \\
s^2Y(s) + 4sY(s) + 4Y(s) &= e^{-3s} + s + 4 \\
(s^2 + 4s + 4)Y(s) &= e^{-3s} + s + 4 \\
Y(s) &= \frac{e^{-3s} + s + 4}{s^2 + 4s + 4} \\
Y(s) &= \frac{e^{-3s} + s + 4}{(s+2)^2} \\
Y(s) &= \frac{e^{-3s}}{(s+2)^2} + \frac{s}{(s+2)^2} + \frac{4}{(s+2)^2}
\end{aligned}$$

We will need the inverse Laplace transform of $\frac{s}{(s+2)^2}$ so first we use partial fractions:

$$\begin{aligned}
\frac{s}{(s+2)^2} &= \frac{A}{s+2} + \frac{B}{(s+2)^2} \\
s &= A(s+2) + B \\
s &= As + 2A + B
\end{aligned}$$

Giving the two equations

$$\begin{aligned}
1 &= A \\
0 &= 2A + B
\end{aligned}$$

Thus $A = 1$, $B = -2$. Therefore $\frac{s}{(s+2)^2} = \frac{1}{s+2} - \frac{2}{(s+2)^2}$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s+2} - \frac{2}{(s+2)^2}\right\} \\
&= e^{-2t} - 2te^{-2t}.
\end{aligned}$$

Applying rules 11 and 13 from Table 6.2.1 we can conclude that

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{Y(s)\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+2)^2} + \frac{s}{(s+2)^2} + \frac{4}{(s+2)^2}\right\} \\
&= u_3(t)(t-3)e^{-2(t-3)} + e^{-2t} - 2te^{-2t} + 4te^{-2t} \\
&= \boxed{u_3(t)(t-3)e^{-2(t-3)} + e^{-2t} + 2te^{-2t}.}
\end{aligned}$$