Practice Midterm 1 Solutions – Math 2415

1. Decide if the following statements are TRUE or FALSE. **You do NOT need to justify your answers.**

(a) (1 point) If the functions φ and ψ are solutions to the differential equation

$$y'' + ay = 0$$

then $\varphi - 2\psi$ is a solution to the differential equation as well.

> **Solution: T** (The set of solutions to linear homogeneous differential equations form a vector space.)

(b) (1 point) If either of the functions p or g is not differentiable at $t = t_0$ then the initial value problem

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0$$

cannot have a unique solution on any open interval I containing t_0.

> **Solution: F** (The initial value problem will have a unique solution as long as p and g are continuous. For example, set t_0, $p(t) = |t|$, $g(t) = |t|$)

(c) (1 point) It is possible for $\varphi(t) = e^t$ and $\psi(t) = 1$ to be solutions to the same first order differential equation

$$y' = f(y, t)$$

where f and $\frac{\partial f}{\partial y}$ are continuous on the entire (y, t)-plane.

> **Solution: F** (This would violate the uniqueness of the solution with initial value $y(0) = 1$)

(d) (1 point) Let p and q be continuous on $\mathbf{R}$. Let y_1 and y_2 be solutions to the differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

If the Wronskian determinant $W(y_1, y_2)(t_0)$ is 0 at t_0 then

$$W(y_1, y_2)(t) = 0$$

for all $t \in \mathbf{R}$.

> **Solution: T** (On an interval of continuity of p and q the Wronskian of two solutions is either always 0 or never 0.)

2. Give examples of the following. Be as explicit as possible. **You do NOT need to justify your answers.**

(a) (2 points) Give an example of a first order differential equation which is separable but not linear.

> **Solution:** $\boxed{\frac{dy}{dx}y^2 = 1 + x}$

(b) (2 points) Give an example of a nonhomogeneous 3rd order linear differential equation with **constant** coefficients.

> **Solution:** $\boxed{y''' + y' - y = 3}$

(c) (2 points) Give an example of a first order autonomous differential equation with exactly two equilibrium solutions.

Solution: $\boxed{\frac{dy}{dt} = (y-2)(y+3)}$

3. (5 points) Compute the Wronskian for $y_1(t) = t^2$, $y_2(t) = \frac{1}{t}$

Solution:

$$\begin{aligned} W(y_1, y_2)(t) &= \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \\ &= \begin{vmatrix} t^2 & \frac{1}{t} \\ 2t & \frac{-1}{t^2} \end{vmatrix} \\ &= t^2 \cdot \left(\frac{-1}{t^2}\right) - 2t \cdot \left(\frac{1}{t}\right) \\ &= -1 - 2 \\ &= \boxed{-3} \end{aligned}$$

4. (5 points) Find the integrating factor for the first order linear equation

$$ty' - 6y - t^2 + 1 = 0.$$

Solution: In standard format this equation is

$$y' - \frac{6}{t} \cdot y = \frac{t^2 - 1}{t}$$

Thus for $t > 0$ an integrating factor is

$$\begin{aligned} \mu(t) &= e^{\int_1^t -\frac{6}{s}\ \mathsf{d}s} \\ &= e^{-6 \ln t} \\ &= \boxed{t^{-6}.} \end{aligned}$$

For $t < 0$ an integrating factor is

$$\begin{aligned} \mu(t) &= e^{\int_{-1}^t -\frac{6}{s}\ \mathsf{d}s} \\ &= e^{-6 \ln(-t)} \\ &= (-t)^{-6} \\ &= \boxed{t^{-6}.} \end{aligned}$$

5. Find general solutions for the following differential equations

(a) (5 points) $y' = (1 + y^2)e^x$.

Solution: This is a first order nonlinear separable equation.

$$\begin{aligned}\frac{dy}{dx} &= (1+y^2)e^x \\ \frac{1}{1+y^2}\cdot\frac{dy}{dx} &= e^x \\ \int \frac{1}{1+y^2}\cdot\frac{dy}{dx}\,dx &= \int e^x\,dx \\ \int \frac{1}{1+y^2}\cdot dy &= \int e^x\,dx \\ \arctan y &= e^x + C \\ y &= \boxed{\tan(e^x + C)}\end{aligned}$$

(b) (5 points) $y'' - 4y' + 4y = 0$.

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$\begin{aligned}r^2 - 4r + 4 &= 0 \\ (r-2)^2 &= 0.\end{aligned}$$

Thus $r = 2$ is a double root.

The general solution is therefore

$$\boxed{y = c_1 e^{2t} + c_2 t e^{2t}.}$$

(c) (5 points) $y'' - 5y' - 6y = 0$.

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$r^2 - 5r - 6 = 0.$$

which has roots

$$r = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-6)}}{2(1)} = \frac{5 \pm \sqrt{49}}{2} = -1 \text{ or } 6$$

The general solution is therefore

$$\boxed{y = c_1 e^{-t} + c_2 e^{6t}.}$$

(d) (5 points) $y'' - 4y' + 13y = 0$.

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$r^2 - 4r + 13 = 0.$$

which has roots

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} = \frac{4 \pm \sqrt{-36}}{2} = 2 \pm 3i$$

The general solution is therefore

$$\boxed{y = c_1 e^{2t}\cos 3t + c_2 e^{2t}\sin 3t.}$$

(e) (5 points) $y' - t^3 y + 1 = 0.$

Solution: This is a first order linear equation. In standard form it is

$$y' - t^3 y = -1.$$

An integrating factor is therefore

$$\begin{aligned}\mu(t) &= e^{\int_0^t -s^3 \text{ ds}} \\ &= e^{\left(-\frac{t^4}{4}\right)}.\end{aligned}$$

The general solution is then

$$y(t) = \frac{y_0 + \int_0^t -e^{-\frac{s^4}{4}}\, ds}{e^{-\frac{t^4}{4}}}.$$

(This is the simplest form we can give for this solution. You can expect the solution to be further simplifiable on the midterm.)

6. Find solutions for the following initial value problems

(a) (5 points) $y'' + 3y' = 0, \quad y(0) = 3, \quad y'(0) = 1.$

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$\begin{aligned} r^2 + 3r &= 0 \\ (r+3)r &= 0.\end{aligned}$$

Thus $r_1 = -3$ and $r_2 = 0$ are roots.
The general solution is therefore

$$y(t) = c_1 e^{-3t} + c_2 e^{0t} = c_1 e^{-3t} + c_2$$

and

$$y'(t) = -3c_1 e^{-3t}.$$

The initial conditions give the equations

$$\begin{aligned} 3 = y(0) &= c_1 e^{-3\cdot 0} + c_2 = c_1 + c_2 \\ 1 = y'(0) &= -3c_1 e^{-3\cdot 0} + c_2 = -3c_1.\end{aligned}$$

Thus

$$c_1 = -\frac{1}{3}$$

$$c_2 = 3 - c_1 = \frac{10}{3}$$

The solution to the IVP is therefore

$$\boxed{y(t) = -\frac{1}{3}e^{-3t} + \frac{10}{3}.}$$

(b) (5 points) $y'' + 6y' + 10 = 0, \quad y(0) = 1, \quad y'(0) = 0.$

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$r^2 + 6r + 10 = 0$$

which has roots

$$r = \frac{-6 \pm \sqrt{6^2 - 4(1)(10)}}{2(1)} = \frac{-6 \pm \sqrt{-4}}{2} = -3 \pm i.$$

The general solution is therefore

$$\begin{aligned} y(t) &= c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t \\ &= e^{-3t}(c_1 \cos t + c_2 \sin t) \end{aligned}$$

and

$$\begin{aligned} y'(t) &= -3e^{-3t}(c_1 \cos t + c_2 \sin t) + e^{-3t}(-c_1 \sin t + c_2 \cos t) \\ &= e^{-3t}((-3c_1 + c_2)\cos t + (-c_1 - 3c_2)\sin t). \end{aligned}$$

The initial conditions give the equations

$$\begin{aligned} 1 &= y(0) = c_1 e^{-3\cdot 0} \cos 0 + c_2 e^{-3\cdot 0} \sin 0 = c_1 \\ 0 &= y'(0) = e^{-3\cdot 0}((-3c_1 + c_2)\cos 0 + (-c_1 - 3c_2)\sin 0) = -3c_1 + c_2 \end{aligned}$$

Thus

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 3c_1 = 3 \end{aligned}$$

The solution to the IVP is therefore

$$\boxed{y(t) = e^{-3t} \cos t + 3e^{-3t} \sin t.}$$

(c) (5 points) $ty' + y = 6, \quad y(1) = 3.$

Solution: This is a first order linear equation. In standard form it is

$$y' + \frac{1}{t}y = \frac{6}{t}.$$

For $t > 0$ An integrating factor is therefore

$$\begin{aligned}\mu(t) &= e^{\int_1^t \frac{1}{s}\,\mathrm{d}s} \\ &= e^{\ln t} \\ &= t\end{aligned}$$

The solution of the IVP is therefore

$$\begin{aligned}y(t) &= \frac{3 + \int_1^t s \cdot \frac{6}{s}\,ds}{t} \\ &= \frac{3 + \int_1^t 6\,ds}{t} \\ &= \boxed{\frac{6t-3}{t}.}\end{aligned}$$

(d) (5 points) $y'' + 8y' + 20y = 0, \quad y(0) = 6, \quad y'(0) = -1.$

Solution: This is a second order linear homogeneous equation with constant coefficients. The charateristic equation is

$$r^2 + 8r + 20 = 0$$

which has roots

$$r = \frac{-8 \pm \sqrt{8^2 - 4(1)(20)}}{2(1)} = \frac{-8 \pm \sqrt{-16}}{2} = -4 \pm 2i.$$

The general solution is therefore

$$\begin{aligned}y(t) &= c_1 e^{-4t}\cos 2t + c_2 e^{-4t}\sin 2t \\ &= e^{-4t}(c_1\cos 2t + c_2 \sin 2t)\end{aligned}$$

and

$$\begin{aligned}y'(t) &= -4e^{-4t}(c_1\cos 2t + c_2\sin 2t) + e^{-4t}(-2c_1\sin 2t + 2c_2\cos 2t) \\ &= e^{-4t}((-4c_1 + 2c_2)\cos 2t + (-2c_1 - 4c_2)\sin 2t).\end{aligned}$$

The initial conditions give the equations

$$\begin{aligned}6 &= y(0) = c_1 e^{-4\cdot 0}\cos 2\cdot 0 + c_2 e^{-4\cdot 0}\sin 2\cdot 0 = c_1 \\ -1 &= y'(0) = e^{-4\cdot 0}((-4c_1 + 2c_2)\cos 2\cdot 0 + (-2c_1 - 4c_2)\sin 2\cdot 0) = -4c_1 + 2c_2\end{aligned}$$

Thus

$$c_1 = 6$$
$$c_2 = \frac{-1 + 4c_1}{2} = \frac{-1 + 4 \cdot 6}{2} = \frac{23}{2}$$

The solution to the IVP is therefore

$$\boxed{y(t) = 6e^{-4t} \cos 2t + \frac{23}{2} e^{-4t} \sin 2t.}$$

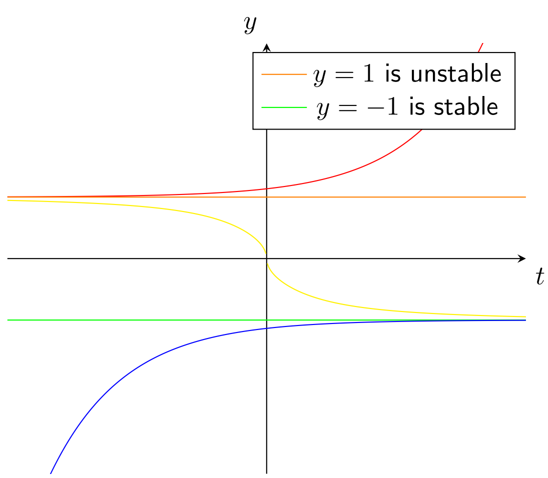
7. (10 points) Sketch solutions to the equation

$$y' = \ln |y|.$$

on the (y, t)-plane. Be sure to include all equilibrium solutions and note whether they are stable, unstable or semistable. Also include a nonequilibrium solution above and below each equilibrium solution.

Solution: This is an autonomous equation. The equilibrium solutions will occur at roots of the equation

$$0 = \ln |y|$$
$$e^0 = e^{\ln |y|}$$
$$1 = |y|$$
$$y = \pm 1$$



8. (10 points) t^2 is a solution to the differential equation

$$3t^2 y'' - 2ty' - 2y = 0.$$

Find the general solution.

Solution: Since we're given a solution t^2 to the differential equation we can look for a solution of the form $t^2u(t)$. Suppose $t^2u(t)$ satisfies the differential equation above.

$$\frac{d}{dt}(t^2u(t)) = 2tu(t) + t^2u'(t)$$
$$\frac{d^2}{dt^2}(t^2u(t)) = 2u(t) + 4tu'(t) + t^2u''(t)$$

thus

$$\begin{aligned}
3t^2(2u(t) + 4tu'(t) + t^2u''(t)) - 2t(2tu(t) + t^2u'(t)) - 2t^2u(t) &= 0 \\
6t^2u(t) + 12t^3u'(t) + 3t^4u''(t) - 4t^2u(t) - 2t^3u'(t) - 2t^2u(t) &= 0 \\
3t^4u''(t) + (12t^3 - 2t^3)u'(t) + (6t^2 - 4t^2 - 2t^2)u(t) &= 0 \\
3t^4u''(t) + 10t^3u'(t) &= 0 \\
u''(t) + \frac{10}{3t}u'(t) &= 0
\end{aligned}$$

This is a first order linear equation for the function u' with integrating factor

$$\mu(t) = e^{\int_1^t \frac{10}{3s}\,ds} = e^{\frac{10}{3}\ln t} = t^{10/3}$$

Thus

$$u'(t) = \frac{C + \int_1^t s^{10/3} \cdot 0\,ds}{t^{10/3}} = Ct^{-10/3}$$
$$u(t) = \int u'(t)\,dt = \int Ct^{-10/3}\,dt = C_1t^{-7/3} + C_2$$

Therefore the general solution is

$$y(t) = (C_1t^{-7/3} + C_2)t^2 = \boxed{C_1t^{-1/3} + C_2t^2}$$

9. (10 points) A forest has the capacity to support 1000 elk but the elk will die off if fewer than 20 are present. The initial population is 100 and at that population size the growth rate is 20% per year. Give a model for logistic growth with a threshold for this population. Sketch the solution to this initial value problem.

Solution: The basic model for logistic growth with capacity $K = 1000$ and threshold $T = 20$ is

$$y' = -r\left(1 - \frac{y}{20}\right)\left(1 - \frac{y}{1000}\right)y$$

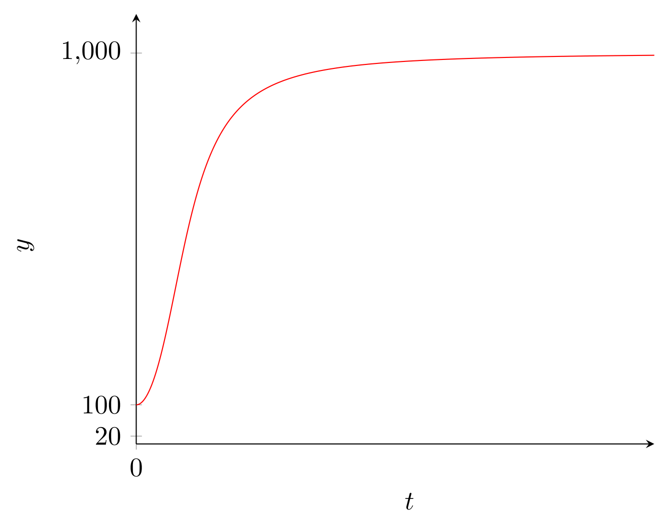
Furthermore, when $y = 100$ we know that $y'/y = 0.2 = \frac{1}{5}$. Thus we may solve for r to get

$$y' = -r\left(1 - \frac{y}{20}\right)\left(1 - \frac{y}{1000}\right)y$$
$$\frac{y'}{y} = -r\left(1 - \frac{y}{20}\right)\left(1 - \frac{y}{1000}\right)$$

$$
\begin{aligned}
\frac{1}{5} &= -r\left(1-\frac{100}{20}\right)\left(1-\frac{100}{1000}\right) \\
\frac{1}{5} &= -r(-4)\frac{9}{10} \\
r &= \frac{1}{18}
\end{aligned}
$$

Thus our population model is

$$
\boxed{y' = -\frac{1}{18}\left(1-\frac{y}{20}\right)\left(1-\frac{y}{1000}\right)y.}
$$



10. (10 points) Suppose that the force of drag on a ball of mass m falling through the air were proportional to the velocity v of the ball. Set up a model for the velocity of the ball falling through the air if the acceleration of gravity is g. Find the velocity as a function of time if the initial velocity is 0.

Solution: At any given time we have two forces acting on the ball. The downward force of gravity with magnitude F_g and the upward force of drag with magnitude F_d. The force of gravity will be considered to be constant

$$F_g = mg.$$

The force of drag depends on the downward velocity v of the ball

$$F_d = kv.$$

The total downward force on the ball is therefore $F = F_g - F_d$. Newton's Second Law asserts that $F = ma$. Thus the acceleration $a = \frac{dv}{dt}$ of the ball is

$$
\begin{aligned}
\frac{dv}{dt} &= \frac{F}{m} \\
&= \frac{F_g - F_d}{m}
\end{aligned}
$$

$$= \frac{mg - kv}{m}$$
$$= g - \frac{kv}{m}$$

Our model for the velocity of the ball is therefore

$$\boxed{\frac{dv}{dt} = g - \frac{k}{m} \cdot v(t), \quad v(0) = 0}$$

where $v(t)$ is the velocity of the ball at time t, g is the acceleration of gravity, m is the mass of the ball, and k is a constant that depends on the viscosity of the atmosphere and size of the ball.

This is a first order linear equation with integrating factor

$$\mu(t) = e^{\int_0^t \frac{k}{m}\, \mathrm{d}s}$$
$$= e^{kt/m}$$

and solution

$$v(t) = \frac{C + \int_0^t e^{ks/m} g\, ds}{e^{kt/m}}$$
$$= Ce^{-kt/m} + \frac{mg}{k}$$

In order to satisfy the intial condition $v(0) = 0$ we solve for C with $t = 0$ to get

$$0 = Ce^0 + \frac{mg}{k}$$
$$C = -\frac{mg}{k}$$

Thus the velocity at time t is

$$\boxed{v(t) = -\frac{mg}{k} e^{-kt/m} + \frac{mg}{k}}$$

11. (10 points) Show that the functions $y_1(t) = t$ and $y_2(t) = \sin(t)$ cannot both be solutions to a single differential equation of the form

$$y'' + p(t)y' + q(t)y = 0$$

where p and q are continuous on $\mathbf{R}$.

Solution: The Wronskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}$$

$$= \begin{vmatrix} t & \sin t \\ 1 & \cos t \end{vmatrix}$$

$$= t\cos t - \sin t$$

The Wronskian is either always 0 or always nonzero for two solutions of the same homogeneous second order linear equation. Notice that $W(y_1, y_2)(0) = 0\cos 0 - \sin 0 = 0$ but $W(y_1, y_2)(\pi) = \pi\cos\pi - \sin\pi = -\pi$. Thus y_1 and y_2 cannot be solutions to the same equation of the the form above.