Math 3345
Fundamentals of Higher Mathematics

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Course Info

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office hours Mondays and Wednesdays 10:10am-11am & 1pm-1:45pm?

Reading for Wednesday, January 15

pgs. 10-14

HW1 and HW2 now due Wednesday, January 15

HW3 Due Friday, January 17

- Section 2 Exercises: 10, 11, 12

Proofs using words

Theorem 21 (Distributive laws)

1. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$
2. $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

- Note: Truth table proof should have $2^3 = 8$ lines.

Proof of 2.

Suppose $P \vee (Q \wedge R)$. Then at least one of P and $(Q \wedge R)$ is true.
Case I: Suppose P is true. Then $P \vee Q$ is true and $P \vee R$ is true. Hence $(P \vee Q) \wedge (P \vee R)$ is true.
Case II: Suppose $(Q \wedge R)$ is true. Then both Q and R must be true. Thus $P \vee Q$ is true and $P \vee R$ is true. Hence $(P \vee Q) \wedge (P \vee R)$ is true.

Proofs using words

Proof of 2 *(continued)*.

Now suppose $P \vee (Q \wedge R)$ is false. Then P is false and $(Q \wedge R)$ is false. For $(Q \wedge R)$ to be false at least one of Q and R must be false.
Case I: Suppose Q is false. Then $P \vee Q$ is false. Hence $(P \vee Q) \wedge (P \vee R)$ is false.
Case II: Suppose R is false. Then $P \vee R$ is false. Hence $(P \vee Q) \wedge (P \vee R)$ is false.
We have shown that when $P \wedge (Q \vee R)$ is true $(P \vee Q) \wedge (P \vee R)$ is true and when $P \wedge (Q \vee R)$ is false $(P \vee Q) \wedge (P \vee R)$ is false. Thus

$$P \wedge (Q \vee R) \equiv (P \vee Q) \wedge (P \vee R).$$

□

- Second half of proof in book is different.
- Both styles are correct.
- Use your favorite (but not both at once!)

Statements involving variables

Example 22

- Consider the statement

$$P(x, y) = \text{“}x > y \Rightarrow x = y.\text{”}$$

- This statment seems to be obviously false, but recall its truth depends on values of x and y.
- For what values of x and y is $P(x, y)$ true?
- For example $P(1, 0) = $ “$1 > 0 \Rightarrow 1 = 0$.”

$1 > 0$	$1 = 0$	$1 > 0 \Rightarrow 1 = 0$
T	F	F

- So $P(1, 0)$ is false as expected.

Statements involving variables

Example 22 *(continued)*

- But $P(0, 1) = $ “$0 > 1 \Rightarrow 0 = 1$.”

$0 > 1$	$0 = 1$	$0 > 1 \Rightarrow 0 = 1$
F	F	T

- So $P(0, 1)$ is true!
- There are 3 possibilities for the numbers $x, y \in \mathbf{R}$.
- **Case I:** Suppose $x, y \in \mathbf{R}$ and $x < y$ is true. Then we have the following truth table:

$x > y$	$x = y$	$x > y \Rightarrow x = y$
F	F	T

Statements involving variables

Example 22 *(continued)*

- **Case II:** Suppose $x, y \in \mathbf{R}$ and $x = y$ is true. Then we have the following truth table:

$x > y$	$x = y$	$x > y \Rightarrow x = y$
F	T	T

- **Case III:** Suppose $x, y \in \mathbf{R}$ and $x > y$ is true. Then we have the following truth table:

$x > y$	$x = y$	$x > y \Rightarrow x = y$
T	F	F

- By Cases I-III $\boxed{P(x, y) \text{ is true for any } x, y \in \mathbf{R} \text{ with } x \leq y}$.
- This example might be confusing because people casually write "$x > y \Rightarrow x = y$" when they mean "For all $x, y \in \mathbf{R}, x > y \Rightarrow x = y$."

Eliminating operations

- Last time I claimed we don't need an xor operation with following truth table:

P	Q	P xor Q
T	T	F
T	F	T
F	T	T
F	F	F

- That's because I can give a logically equivalent statement that only involves $\neg$, $\wedge$, and $\vee$.
- I claim P xor Q as defined above is logically equivalent to

$$(P \wedge \neg Q) \vee (\neg P \wedge Q).$$

- Give a proof!

Eliminating operations

- xor example above might suggest way to get statement with any given truth table.
- For example, give a stament using only $\neg$, $\wedge$, and $\vee$ with the following truth table:

P	Q	?
T	T	T
T	F	F
F	T	T
F	F	F

- We want our mystery statement to be true in two instances:
  1. when P and Q are both true (and hence $P \wedge Q$ is true).
  2. when P is false and Q is true (and hence $\neg P \wedge Q$ is true).
- Let ? be $(P \wedge Q) \vee (\neg P \wedge Q)$.
- Try to give a statement logically equivalent to $P \Leftrightarrow Q$ using only $\neg$, $\wedge$, and $\vee$.

Eliminating operations

It should now be clear that

Theorem 23

Any logical statment is logically equivalent to one using only atomic statements and the operations $\neg$, $\wedge$, *and* $\vee$.

Actually we can do even better:

Example 24 (Eliminating $\wedge$)

Give a statement logically equivalent to $P \wedge Q$ which involves only the operations $\neg$ and $\vee$.

$$\begin{aligned} P \wedge Q &\equiv \neg\neg(P \wedge Q) \\ &\equiv \neg(\neg P \vee \neg Q). \end{aligned}$$

Thus we really only need $\neg$ and $\vee$ operations.

Tautologies

Definition 25 (Tautology)

A **tautology** is a statement involving undetermined statements P, Q, etc. which is true regardless of the truth value of the undetermined statements.

Example 26

Show that $P \vee \neg P$ is a tautology.

Proof.

Either P is true or P is false.
Case I: Suppose P is true. Then $P \vee \neg P$ is true.
Case II: Suppose P is false. Then $\neg P$ is true. Hence $P \vee \neg P$ is true. □