Math 3345
Fundamentals of Higher Mathematics

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Course Info

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Reading for Friday, January 17
pgs. 15-18

HW4 Due Wednesday, January 22

▶ Section 2 Exercises: 7a, 13, 14
Warm-up Problems

Problem 27

Solution to Exercise 5b

Problem 28

For what real numbers \( x \) is the statement

\[ P(x) = 1 \leq x < 2 \]

false?

Note

\( 1 \leq x < 2 \) means \( (1 \leq x) \land (x < 2) \).

Solution to Problem 28

\[ \neg(1 \leq x < 2) \equiv \neg((1 \leq x) \land (x < 2)) \]
\[ \equiv \neg(1 \leq x) \lor \neg(x < 2) \]
\[ \equiv (1 > x) \lor (x \geq 2) \]

Thus \( P(x) \) is false if the real number \( x \) is less than 1 or greater than or equal to 2.
More on implications

**Definition 29 (Converse and Contrapositive)**

Given an implication $P \Rightarrow Q$ the **converse** is the statement $Q \Rightarrow P$ and the **contrapositive** is the statement $\neg Q \Rightarrow \neg P$.

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**Theorem 30**

$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$

**Warning**

A statement is not logically equivalent to its converse.

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More on implications

**Example 31**

- Let $Q$ be the statement “If today is Monday then tomorrow is Tuesday.”
- Then converse of $Q$ is “If tomorrow is Tuesday then today is Monday.”
- The contrapositive of $Q$ is “If tomorrow is not Tuesday then today is not Monday.”
- In this case all statements are true.
More on implications

**Example 32**

- Let $P(x)$ be the statement “If $x$ is an integer then $x$ is a real number.”
- Then converse of $P(x)$ is $Q(x) =$ “If $x$ is a real number then $x$ is an integer.”
- The contrapositive of $P(x)$ is $R(x) =$ “If $x$ is not a real number then $x$ is not an integer.”
- In this case $P(x)$ and $R(x)$ are true no matter the value of $x$.
- But the converse $Q(x)$ is only true if $x$ is an integer or $x$ is not a real number.

**Example 33**

- Let $R$ be the statement “$(1 + 2 = 5) \land (1 < 2)$”.
- Then $R$ has no converse or contrapositive since it is not an implication.

More on tautologies

**Example 34 (Tautologies)**

The following statements are all tautologies:

1. $P \lor \neg P$  Law of the excluded middle
2. $P \Rightarrow P$
3. $(P \land Q) \Rightarrow P$  Example 2.21 in book
4. $Q \Rightarrow (P \lor Q)$  Exercise 13 (HW4 problem)
5. $(P \land (P \Rightarrow Q)) \Rightarrow Q$  Modus ponens
6. $Q \Rightarrow (P \Rightarrow Q)$

All the above may be proven with truth tables.
More on tautologies

Definition 35 (Contradiction)

A contradiction is a sentence of the form $Q \land \neg Q$ and hence always false.

Warning

Contradictions are always of this form.

Example 36 (Contradictions)

The following statements are all contradictions

1. $1 = 2 \land 1 \neq 2$
2. $(x$ is even) $\land$ $(x$ is not even) $\land$

All the above may be proven with truth tables.

Conditional proof

Method of conditional proof

In order to prove the conditional statement $A \Rightarrow B$ assume that $A$ is true and under that assumption show that $B$ is true.

Theorem 37

$Q \Rightarrow (P \Rightarrow Q)$ is a tautology.

Proof.

We will use the method of conditional proof.

Assume that $Q$ is true.

Then for any statement $P$ the statement $P \Rightarrow Q$ holds.

Discharging $A1$ we have proven that $Q \Rightarrow (P \Rightarrow Q)$ is true without any assumptions. Thus $Q \Rightarrow (P \Rightarrow Q)$ is a tautology.  \qed
A proof using the method of conditional proof should always look like:

**Theorem 38**

\[ A \Rightarrow B \]

**Proof.**

We will use the method of conditional proof.

\[ A1 \]

Assume \( A \) is true.

Then

\[ \vdots \]

thus \( B \) is true.

Discharging \( A1 \) we have proven that \( A \Rightarrow B \) is true without any assumptions.

**Warning**

Method of conditional proof can only be used to prove conditional statements.