Math 3345
Fundamentals of Higher Mathematics

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Course Info

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Reading for Friday, January 31

pgs. 34-36

HW8 Due Monday, February 3

- Section 3 Exercises: 5, 7, 8

Bonus Problems (Hand in any time before April 18)

Section 2 Exercises 18, 29, 34

Scope of quantifiers

Scope of quantifiers

In a quantified statement such as

$$(\forall x)P(x)$$

the **scope** of the quantifier $(\forall x)$ is the statement $P(x)$. we say that the variable x is **bound** within the statement $P(x)$ and **free** outside of $P(x)$.

Example 50

Underline the scope of the bound variable x in the following statements:

1. $(\forall x \in \mathbf{R})\underline{(x+1>x)}$
2. $(x>5) \wedge (\exists x \in \mathbf{R})\underline{(x+1>x)}$

Note that statement 1 above has no free variables and its truth value is decidable (true) whereas statement 2 above has a free variable x on which its truth value depends.

Generalization of De Morgan's Law

Theorem 51 (Generalized De Morgan's Law)

1. $\neg(\forall x)P(x) \equiv (\exists x)\neg P(x)$
2. $\neg(\exists x)P(x) \equiv (\forall x)\neg P(x)$

Proof.

	$\neg(\forall x)P(x)$ is true
iff	$(\forall x)P(x)$ is false
iff	$P(x)$ is false for at least on value of x.
iff	$\neg P(x)$ is true for at least on value of x.
iff	$(\exists x)\neg P(x)$ is true.

□

Generalization of De Morgan's Law

Theorem 52 (Generalized Distributive Rule)

If P is a statement that does not involve the variable x then

1. $P \wedge (\exists x)Q(x) \equiv (\exists x)[P \wedge Q(x)]$
2. $P \vee (\forall x)Q(x) \equiv (\forall x)[P \vee Q(x)]$
3. $P \wedge (\forall x)Q(x) \equiv (\forall x)[P \wedge Q(x)]$
4. $P \vee (\exists x)Q(x) \equiv (\exists x)[P \vee Q(x)]$

- These theorems relate to original De Morgan and Distributive Laws if one views $(\forall x)P(x)$ as the "anding" together of the statements $P(x)$ for all x's in the universe of discourse and one views $(\exists x)P(x)$ as the "oring" together of the statements $P(x)$ for all x's in the universe of discourse.

Example 53

1. Let S be the set of all people.

$$\begin{aligned} & \neg(\text{Everyone makes mistakes.}) \\ \equiv\ & \neg(\forall x \in S)(x \text{ makes mistakes.}) \\ \equiv\ & (\exists x \in S)\neg(x \text{ makes mistakes.}) \\ \equiv\ & (\exists x \in S)(x \text{ does not makes mistakes.}) \\ \equiv\ & \text{There is at least one person who does not make mistakes.} \end{aligned}$$

2. Decide if the statement

$$\neg(\forall x \in \mathbf{R})[(x > 2) \vee (x^2 > 4)]$$

is true.

$$\begin{aligned} & \neg(\forall x \in \mathbf{R})[(x > 2) \vee (x^2 > 4)] \\ \equiv\ & (\exists x \in \mathbf{R})\neg[(x > 2) \vee (x^2 > 4)] \\ \equiv\ & (\exists x \in \mathbf{R})[\neg(x > 2) \wedge \neg(x^2 > 4)] \\ \equiv\ & (\exists x \in \mathbf{R})[(x \leq 2) \wedge (x^2 \leq 4)] \end{aligned}$$

This statement is true (example: $x = 0$)

Nested quantifiers

- In a quantified statement $(\forall x)P(x)$ the statement $P(x)$ can itself be a quantified statement.

Example 54 (Order of quantifiers matters)

1. $(\forall x \in \mathbf{R})(\exists y \in \mathbf{N})x < y$. means For every real number x there is a natural number y which is bigger. This is true.
2. $(\exists y \in \mathbf{N})(\forall x \in \mathbf{R})x < y$. means There is a natural number y which is bigger than every real number x. This is false.

Lecture 9 - 1/29/2014 Scope of quantifiers Generalization of De Morgan's Law Nested quantifiers

Nested quantifiers

Example 55

Give a sentence which asserts that there are two people in class with the same birthday. Let C be the set of people in class and let $b(x)$ denote the birthday of person x.

$$(\exists x \in C)(\exists y \in C)\{[b(x) = b(y)] \wedge x \neq y\}$$

Note that $(\exists x \in C)(\exists y \in C)b(x) = b(y)$ is true of **any** class with at least one person.

- Remember, different variables can represent the same value!

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Uniqueness quantifier

Uniqueness quantifier

The statement

$$(\exists! x)P(x)$$

means "There exists a unique x such that $P(x)$ is true."

Example 56

1. $$(\exists! x \in \mathbf{R})x + 5 = 7.$$

   is **true** since $x = 2$ is the only solution.

2. $$(\exists! x \in \mathbf{R})x^2 = 5.$$

   is **false** since $x = \sqrt{5}$ and $x = -\sqrt{5}$ are two different solutions.

3. $$(\exists! x \in \mathbf{R})x^2 = -9.$$

   is **false** since there are no solutions.

Uniqueness quantifier

The statement

$$(\exists! x)P(x)$$

is logically equivalent to the statement

$$(\exists x)\{P(x) \wedge (\forall y)[P(y) \Rightarrow (y = x)]\}.$$

Thus our new quantifier $\exists!$ does not expand the set of logical meanings our sentences can have.