Math 3345
Fundamentals of Higher Mathematics

Nathan Broaddus

Ohio State University

January 31, 2014

Course Info

Instructor - Nathan Broaddus

webpage https://people.math.osu.edu/broaddus.9/3345
office hours Mondays and Wednesdays 10:10am-11am & 1pm-1:45pm
office MW650

Reading for Monday, February 3

pgs. 37-40

HW9 Due Monday, February 10

- Section 3 Exercises: 10bcf, 11, 14dfj

Bonus Problems (Hand in any time before April 18)

Section 2 Exercises 18, 29, 34
Section 3 Exercise 13

Nested quantifiers

Nested quantifiers must use different variables

Nested quantifiers must use different variables!

$$(\forall x \in \mathbf{R})(\exists x \in \mathbf{R})x = 5$$

is nonsense. But

$$[(\forall x \in \mathbf{R})x^2 \geq 0] \wedge [(\exists x \in \mathbf{R})2x = 5]$$

is ok.

Proofs

Tips for proofs

1. **Definitions** tell you where to start and end a proof.
2. State **all** assumptions.
3. Use propositional calculus to combine assumptions and definitions to work towards conclusion of proof.
4. Only use definitions from book or class.

Even and odd

Forget everything you know about the words even and odd.

Definition 57 (Even number)

x is an **even number** if there exists $k \in \mathbf{Z}$ such that $x = 2k$.

Definition 58 (Odd number)

x is an **odd number** if there exists $k \in \mathbf{Z}$ such that $x = 2k + 1$.

Example 59

1. 10 is an even number since $10 = 2 \cdot 5$ and 5 is an integer.
2. 21 is and odd number since $21 = 2 \cdot 10 + 1$ and 10 is an integer.

Even and odd

Theorem 60

If x is an odd number and y is an odd number then $x + y$ is an even number.

Proof.

We will proceed by the method of conditional proof. Suppose that x is an odd number and y is an odd number. Then there exists $k \in \mathbf{Z}$ such that $x = 2k + 1$ and there exists $\ell \in \mathbf{Z}$ such that $y = 2\ell + 1$. Hence

$$\begin{aligned} x + y &= (2k + 1) + (2\ell + 1) \\ &= 2k + 2\ell + 2 \\ &= 2(k + \ell + 1) \end{aligned}$$

k, ℓ and 1 are integers so $k + \ell + 1$ is an integer. Hence $x + y$ is an even number. □

Even and odd

- We have **not** defined odd to mean "not even"
- In fact this is not the right way to define odd since there are numbers (like $\frac{2}{3}$) which are not even or odd.
- With our definitions it is not clear that every integer is even or odd. (proof later by induction)
- It is also not clear that an integer cannot be both even and odd. (proof below)

We need the following fact about integers:

Lemma 61

If x is an integer then $x \leq 0$ or $x \geq 1$.

Theorem 62

If x is an integer then x cannot be both an even number and an odd number.

Even and odd

Proof of Theorem 62.

We will proceed by the method of proof by contradiction. Suppose that it is not true that

$$(\forall x \in \mathbf{Z})\neg[(x \text{ even}) \wedge (x \text{ odd})]$$

Then the negation

$$\neg(\forall x \in \mathbf{Z})\neg[(x \text{ even}) \wedge (x \text{ odd})]$$
$$\equiv (\exists x \in \mathbf{Z})[(x \text{ even}) \wedge (x \text{ odd})]$$

must be true. Let $x \in \mathbf{Z}$ be an even and odd number. Then there is $k \in \mathbf{Z}$ such that $x = 2k$ and there is $\ell \in \mathbf{Z}$ such that $x = 2\ell + 1$.

Even and odd

Proof of Theorem 62 *(continued.)*

Therefore

$$2k = 2\ell + 1$$
$$\Rightarrow 2k - 2\ell = 1$$
$$\Rightarrow 2(k - \ell) = 1$$

Let $m = k - \ell$. Note that $k, \ell \in \mathbf{Z}$ so $m \in \mathbf{Z}$ and $2m = 1$. By Lemma 61 above $m \in \mathbf{Z}$ so $m \leq 0$ or $m \geq 1$. Multiplying these inequalities by 2 we get that $2m \leq 0$ or $2m \geq 2$. Hence $2m \neq 1$. We have shown that $2m = 1$ and $2m \neq 1$. This is a contradiction. Thus we have proven that

$$(\forall x \in \mathbf{Z})\neg[(x \text{ even}) \wedge (x \text{ odd})].$$

□

Even and odd

Scope of variables in proofs

- In proof above we introduced a variable $k \in \mathbf{Z}$ such that $x = 2k$ and $\ell \in \mathbf{Z}$ such that $x = 2\ell + 1$.
- Why not use k for both purposes?
- Because it would not be clear if $x = 2k$ or $x = 2k + 1$.
- The **scope** of such a variable is the rest of the proof.
- Therefore they cannot be reused for a different purpose!