Math 3345
Fundamentals of Higher Mathematics

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Course Info

Reading for Monday, February 24

pgs. 62-64

HW14 Due Wesnesday, February 26

- Section 4 Exercises 19ab, 26a
- Section 5 Exercise 2

Bonus Problems (Hand in any time before April 18)

Section 2 Exercises 18, 29, 34
Section 3 Exercise 13
Section 4 Exercises 16, 24

Warm-up Problems

Problem 1

Solution to Section 4 Exercise 17a:

If x is a rational number and c is an integer such that $x^3 = c$ then x is an integer.

Binomial coefficients

Definition 2 (Binomial coefficients)

Consider the following three rules:

1. $$\binom{0}{0} = 1$$
2. For all $n \in \mathbf{N}$
$$\binom{n}{0} = \binom{n}{n} = 1$$
3. For all $n, k \in \mathbf{N}$ such that $1 \leq k \leq n$
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

For each $n, k \in \omega$ such that $0 \leq k \leq n$ the **binomial coefficent** $\binom{n}{k}$ (read "n choose k") satisfies rules 1-3 above.

Binomial coefficients

Example 3

Use Definition 2 above to compute $\binom{4}{2}$.

$$\begin{aligned}\binom{4}{2} &= \binom{3}{2} + \binom{3}{1} \\ &= \binom{2}{2} + \binom{2}{1} + \binom{2}{1} + \binom{2}{0} \\ &= 1 + 2\binom{2}{1} + 1 \\ &= 1 + 2\left[\binom{1}{1} + \binom{1}{0}\right] + 1 \\ &= 1 + 2(1+1) + 1 \\ &= 6\end{aligned}$$

Binomial coefficients

Why is $\binom{n}{k}$ called n choose k?

Example 4

List the subsets of the set $\{1, 2, 3, 4\}$ with exactly 2 elements:

$$\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}$$

Notice that there are $6 = \binom{4}{2}$ such sets.

In Section 14 we will prove:

Theorem 5

Let n and k be whole numbers such that $0 \leq k \leq n$. A set with n elements has $\binom{n}{k}$ subsets with k elements.

Thus there are $\binom{n}{k}$ ways to "choose" k elements from a set with n elements.

Binomial coefficients

Why is $\binom{n}{k}$ called a binomial coefficient?

Example 6

1. Compute $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, and $\binom{3}{3}$

$$\binom{3}{0} = 1, \quad \binom{3}{1} = \binom{2}{1} + \binom{2}{0} = \binom{1}{1} + \binom{1}{0} + \binom{2}{0} = 3,$$
$$\binom{3}{2} = \binom{2}{2} + \binom{2}{1} = \binom{2}{2} + \binom{1}{1} + \binom{1}{0} = 3, \quad \binom{3}{3} = 1$$

2. Give the cube of the binomial $(a+b)$:

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b) \\ &= (a^2 + 2ab + b^2)(a+b) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

Notice that $(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$

Binomial coefficients

Example 6 suggests a theorem. What should it say?

Theorem 7 (The Binomial Theorem)

For all $n \in \omega$ and all $a, b \in \mathbf{R}$

$$\begin{aligned}(a+b)^n &= \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n \\ &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k}b^k.\end{aligned}$$

Proof of Theorem 7: Future HW problem

Note: Above theorem requires that $x^0 = 1$ for all $x \in \mathbf{R}$. In particular $0^0 = 1$. This is actually standard.