# Math 3345
# Fundamentals of Higher Mathematics

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February 26, 2014





## Course Info

### Midterm 2 Wednesday, March 5 in class

Midterm 2 will cover Sections 1-5.

# Warm-up Problems

## Problem 1

*Solution to Section 4 Exercise 19a:*

*If $x$ is a rational number such that $x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$ where $n \in \mathbf{N}$ and $c_0, c_1, \cdots, c_{n-1} \in \mathbf{Z}$, then $x$ is an integer.*

## Problem 2

*Solution to Section 4 Exercise 22:*

*Let $p \in \{2, 3, 4, \cdots\}$. Suppose that for all $x, y \in \mathbf{Z}$, if $p$ divides $xy$ then $p$ divides $x$ or $p$ divides $y$.*





# Binomial coefficients

You've probably seen a different definition of $\binom{n}{k}$

## Definition 3 (Factorials)

$0! = 1$ and for each $n \in \mathbf{N}$

$$\begin{aligned} n! &= 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n \\ &= \prod_{k=1}^{n} k. \end{aligned}$$

You've probably seen the binomial coefficients defined via the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Our definition of $\binom{n}{k}$ is different so for us this is a theorem that must be proven.

# Binomial coefficients

## Theorem 4

*For all $n, k \in \omega$ such that $0 \leq k \leq n$*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

## Proof.

Let $P(n)$ be the statement:

$$(\forall k \in \omega)\left((0 \leq k \leq n) \Rightarrow \left[\binom{n}{k} = \frac{n!}{k!(n-k)!}\right]\right)$$

We will prove that $P(n)$ is true for all $n \in \omega$ by induction.
Base Case: Suppose $n = 0$ and $k \in \omega$ satisfies $0 \leq k \leq n$. Then $k = 0$ so $\binom{n}{k} = \binom{0}{0} = 1 = \frac{0!}{0!(0-0)!} = \frac{n!}{k!(n-k)!}$. Thus $P(0)$ is true.



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# Binomial coefficients

## Proof of Theorem 4 *(continued)*.

Inductive Step: Suppose that $n \in \omega$ and $P(n)$ is true and $k \in \omega$ satisfies $0 \leq k \leq n+1$. Then $k = 0$, $k = n+1$ or $1 \leq k \leq n$.
Case 1: Suppose $k = 0$.
Then $\binom{n+1}{k} = \binom{n+1}{0} = 1 = \frac{(n+1)!}{(n+1)!} = \frac{(n+1)!}{0!(n+1-0)!} = \frac{(n+1)!}{k!(n+1-k)!}$.
Case 2: Suppose $k = n+1$.
Then $\binom{n+1}{k} = \binom{n+1}{n+1} = 1 = \frac{(n+1)!}{(n+1)!} = \frac{(n+1)!}{(n+1)!(n+1-(n+1))!} = \frac{(n+1)!}{k!(n+1-k)!}$.

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## Proof of Theorem 4 *(continued)*.

Case 3: Suppose $1 \leq k \leq n$. Then

$$\begin{aligned}
\binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1} \\
&= \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!} \\
&= \frac{n!}{k(k-1)!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)(n-k)!} \\
&= \frac{(n-k+1)n!}{(n-k+1)k(k-1)!(n-k)!} + \frac{kn!}{k(k-1)!(n-k+1)(n-k)!} \\
&= \frac{(n-k+1)n! + kn!}{k(k-1)!(n-k+1)(n-k)!} \\
&= \frac{(n+1)n!}{k(k-1)!(n-k+1)(n-k)!} = \frac{(n+1)!}{k!(n+1-k)!}
\end{aligned}$$

Thus by induction $P(n)$ is true for all $n \in \omega$. □