Math 3345
Fundamentals of Higher Mathematics

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Course Info

HW16 Due Friday, March 7

- Section 5 Exercises 12, 13, 18

Midterm 2 Wednesday, March 5 in class

Midterm 2 will cover Sections 1-5.

Warm-up Problems

Problem 1

Solution to Section 4 Exercise 20:

If x is a rational number such that $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0$ where $n \in \mathbf{N}$ and $c_0, c_1, \cdots, c_{n-1}, c_n \in \mathbf{Z}$, then there are $a, b \in \mathbf{Z}$ such that $x = \frac{a}{b}$, a divides c_0 and b divides c_n.

Lecture 19 - 2/28/2014

Division Lemma

In the next proof we will use induction to prove a statement $P(x)$ for all $x \in \mathbf{Z}$.

Lemma 2

Let $d \in \mathbf{N}$. Then for all $x \in \mathbf{Z}$, there are $q, r \in \mathbf{Z}$ with $0 \leq r < d$ such that $x = qd + r$.

Proof.

Given $d \in \mathbf{N}$ let $P(x)$ be the statement:

$$(\exists r, q \in \mathbf{Z})[(0 \leq r < d) \wedge (x = qd + r)]$$

We will prove that $P(x)$ is true for all $x \in \mathbf{Z}$ by induction.

Base Case: $0 = 0d + 0$ and $(0 \leq 0 < d)$ so $P(0)$ is true.

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Division Lemma

Proof of Lemma 2 *(continued)*.

Inductive Step I: Suppose that $x \in \mathbf{Z}$ and $P(x)$ is true.
Then there are $r, q \in \mathbf{Z}$ such that $0 \leq r < d$ and $x = qd + r$.
$r + 1 < d + 1$ so $r + 1 < d$ or $r + 1 = d$.
Case 1: Suppose $r + 1 < d$.
Then $x + 1 = qd + (r + 1)$. Hence $P(x + 1)$ is true.
Case 2: Suppose $r + 1 = d$.
Then $x + 1 = qd + r + 1 = qd + d = (q + 1)d + 0$. Hence $P(x + 1)$ is true.

Division Lemma

Proof of Lemma 2 *(continued)*.

Inductive Step II: Suppose that $x \in \mathbf{Z}$ and $P(x)$ is true.
Then there are $r, q \in \mathbf{Z}$ such that $0 \leq r < d$ and $x = qd + r$.
$r - 1 \geq 0 - 1$ so $r - 1 \geq 0$ or $r - 1 = -1$.
Case 1: Suppose $r - 1 \geq 0$.
Then $x - 1 = qd + (r - 1)$. Hence $P(x - 1)$ is true.
Case 2: Suppose $r - 1 = -1$.
Note that $d \geq 1$ so $0 \leq d - 1 \leq d - 1$ Then $x - 1 = qd + r - 1$
$= qd + d - d - 1 = (q - 1)d + (d - 1)$. Hence $P(x - 1)$ is true.
Now using our Base case, Inductive step I and Inductive step II we have shown that $P(x)$ is true for all $x \in \mathbf{Z}$.

Division Lemma

Lemma 3 (The Division Lemma)

Let $d \in \mathbf{N}$. Then for all $x \in \mathbf{Z}$, there are unique $q, r \in \mathbf{Z}$ with $0 \leq r < d$ such that $x = qd + r$.

Proof.

By Lemma 2

$$(\exists r, q \in \mathbf{Z})[(0 \leq r < d) \wedge (x = qd + r)]$$

We will prove that r and q are **unique**.
Suppose $d \in \mathbf{N}$ and $x \in \mathbf{Z}$ and $r_1, q_1, r_2, q_2 \in \mathbf{Z}$ satisfy $0 \leq r_1 < d$, $0 \leq r_1 < d$, $x = q_1 d + r_1$ and $x = q_2 d + r_2$. We wish to show that $r_1 = r_2$ and $q_1 = q_2$ (This is the meaning of uniqueness). r_1 and r_2 are integers so $r_2 \leq r_1 - 1$, $r_2 = r_1$ or $r_2 \geq r_1 + 1$.

Division Lemma

Proof of Lemma 3 *(continued)*.

Case 1: Suppose $r_2 \leq r_1 - 1$.
Then $r_1 - r_2 \in \mathbf{N}$. $x = q_1 d + r_1$ and $x = q_2 d + r_2$. Hence $q_1 d + r_1 = q_2 d + r_2$. Therefore $r_1 - r_2 = q_2 d - q_1 d = (q_2 - q_1)d$. It follows that the natural number d divides the natural number $r_1 - r_2$. Hence by Remark 4.38 $d \leq r_1 - r_2$. But $r_1 < d$ so $r_1 - r_2 < d - r_2 \leq d$ so $r_1 - r_2 < d$. Hence $d \not\leq r_1 - r_2$ and $d \leq r_1 - r_2$. This is a contradiction.
Case 2: Suppose $r_2 \geq r_1 + 1$.
Then $r_2 - r_1 \in \mathbf{N}$. $x = q_1 d + r_1$ and $x = q_2 d + r_2$. Hence $q_1 d + r_1 = q_2 d + r_2$. Therefore $r_2 - r_1 = q_1 d - q_2 d = (q_1 - q_2)d$. It follows that the natural number d divides the natural number $r_2 - r_1$. Hence by Remark 4.38 $d \leq r_2 - r_1$. But $r_2 < d$ so $r_2 - r_1 < d - r_1 \leq d$ so $r_2 - r_1 < d$. Hence $d \not\leq r_2 - r_1$ and $d \leq r_2 - r_1$. This is a contradiction.
We have eliminated the two other possibilities so we must have that $r_1 = r_2$. Thus $x = q_1 d + r_1$ and $x = q_2 d + r_1$. Hence $q_1 d + r_1 = q_2 d + r_1$. Therefore $(q_1 - q_2)d = r_1 - r_1 = 0$. It follows that $q_1 - q_2) = 0$ or $d = 0$. $d \in \mathbf{N}$ so $d \neq 0$. It follows that $q_1 - q_2 = 0$. Hence $q_1 = q_2$. □