Math 3345
Fundamentals of Higher Mathematics

Nathan Broaddus

Ohio State University

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Course Info

Quiz 3 Friday, March 28 in class

Quiz 3 will cover pgs. 97-107

Intervals in **R**

Definition 1 (Intervals in **R**)

Given $a, b \in \mathbf{R}$ the following subsets of **R** are **open intervals**:

1. $$(a, b) = \{x \in \mathbf{R} |\, a < x < b\}$$
2. $$(a, \infty) = \{x \in \mathbf{R} |\, a < x\}$$
3. $$(-\infty, b) = \{x \in \mathbf{R} |\, x < b\}$$
4. $$(-\infty, \infty) = \mathbf{R}$$

Intervals in **R**

Definition 2 (Intervals in **R**)

Given $a, b \in \mathbf{R}$ the following subsets of **R** are **closed intervals**:

1. $$[a, b] = \{x \in \mathbf{R} |\, a \leq x \leq b\}$$
2. $$[a, \infty) = \{x \in \mathbf{R} |\, a \leq x\}$$
3. $$(-\infty, b] = \{x \in \mathbf{R} |\, x \leq b\}$$
4. $$(-\infty, \infty) = \mathbf{R}$$

Intervals in $\mathbf{R}$

Definition 3 (Intervals in $\mathbf{R}$)

Given $a, b \in \mathbf{R}$ the following subsets of $\mathbf{R}$ are **half-open intervals**:

1. $$(a, b] = \{x \in \mathbf{R} | \, a < x \leq b\}$$
2. $$[a, b) = \{x \in \mathbf{R} | \, a \leq x < b\}$$

Intervals in $\mathbf{R}$

Example 4 (Intervals in $\mathbf{R}$)

1. $$(1, 2) \cup [2, 4] = (1, 4]$$
2. $$(2, 7) \cap [7, 10) = \varnothing$$

Sets of Sets

Definition 5 (Union of a set of sets)

If $\mathcal{A}$ is a set of sets then the **union of** $\mathcal{A}$ is the set

$$\bigcup \mathcal{A} = \{x \mid \exists A \in \mathcal{A}, \text{ such that } x \in A\}$$

Definition 6 (Intersection of a set of sets)

If $\mathcal{A}$ is a **nonempty** set of sets then the **intersection of** $\mathcal{A}$ is the set

$$\bigcap \mathcal{A} = \{x \mid \forall A \in \mathcal{A}, x \in A\}$$

Sets of Sets

Example 7 (Unions and intersections of sets of sets)

1. Let $\mathcal{A} = \{\{1,2\},\{2,3\},\varnothing\}$.

$$\bigcup \mathcal{A} = \{1,2,3\}$$
$$\bigcap \mathcal{A} = \varnothing$$

2. Let $\mathcal{B} = \varnothing$.

$$\bigcup \mathcal{B} = \varnothing$$
$$\bigcap \mathcal{B} \text{ is undefined}$$

3. If $\mathcal{C} = \{1,\{2\}\}$ then $\bigcup \mathcal{C}$ and $\bigcup \mathcal{C}$ are undefined since $\mathcal{C}$ is not a set of sets.

Sets of Sets

Theorem 8

If $\mathcal{A}$ is a nonempty set of sets then

$$(\forall A \in \mathcal{A}) \bigcap \mathcal{A} \subset A \subset \bigcup \mathcal{A}.$$

Proof.

Suppose $\mathcal{A}$ is a nonempty set of sets and $A \in \mathcal{A}$. Assume that $x \in \bigcap \mathcal{A}$.
Then $(\forall A \in \mathcal{A})\, x \in A$. $A \in \mathcal{A}$ so $x \in A$.
Hence $(\forall x)[(x \in \bigcap \mathcal{A}) \Rightarrow (x \in A)]$.
Therefore

$$(\forall A \in \mathcal{A}) \bigcap \mathcal{A} \subset A.$$

□

Sets of Sets

Theorem 9 (Generalized De Morgan's Laws for sets)

Let S be a set and let $\mathcal{A}$ be a nonempty set of sets. Then

1. $S \setminus \bigcup \mathcal{A} = \bigcap \{S \setminus A \mid A \in \mathcal{A}\}$.
2. $S \setminus \bigcap \mathcal{A} = \bigcup \{S \setminus A \mid A \in \mathcal{A}\}$.

Proof of 1.

Let S be a set and let $\mathcal{A}$ be a nonempty set of sets. Given x
$x \in S \setminus \bigcup \mathcal{A}$
iff $x \in S$ and $\neg(x \in \bigcup \mathcal{A})$
iff $x \in S$ and $\neg(\exists A \in \mathcal{A})\, x \in A$
iff $x \in S$ and $(\forall A \in \mathcal{A})\, \neg x \in A$
iff $(\forall A \in \mathcal{A})\, (x \in S$ and $\neg x \in A)$
iff $(\forall A \in \mathcal{A})\, (x \in S \setminus A)$
iff $x \in \bigcap \{(S \setminus A) \mid A \in \mathcal{A}\}$. □

Sets of Sets

Theorem 10 (Generalized distributive rule for sets)

Let S be a set and let $\mathcal{A}$ be a nonempty set of sets. Then

1. $S \cap \bigcup \mathcal{A} = \bigcup \{S \cap A | \, A \in \mathcal{A}\}$.
2. $S \cup \bigcap \mathcal{A} = \bigcap \{S \cup A | \, A \in \mathcal{A}\}$.

The Power Set

Definition 11 (The Power Set)

Let S be a set. The **power set** of S is the set

$$\mathcal{P}(S) = \{A | \, A \subset S\}$$

Example 12 (Power sets)

1. $$\mathcal{P}(\{1,2\}) = \big\{\varnothing, \{1\}, \{2\}, \{1,2\}\big\}$$
2. $$\mathcal{P}(\varnothing) = \big\{\varnothing\big\}$$
3. $$\mathcal{P}(\mathcal{P}(\varnothing)) = \mathcal{P}(\{\varnothing\}) = \big\{\varnothing, \{\varnothing\}\big\}$$

Disjointness

Definition 13

1. Sets A and B are **disjoint** if
$$A \cap B = \varnothing$$
2. If $\mathcal{A}$ is a nonempty set of sets then $\mathcal{A}$ is **pairwise disjoint** if for all $A, B \in \mathcal{A}$
$$A = B \text{ or } A \cap B = \varnothing$$

Example 14

1. The sets $\{1, 2, 3\}$ and $\{4, 5\}$ are disjoint.
2. The set $\{\{1, 2, 3\}, \{6, 7\}, \{7, 8\}\}$ is not pairwise disjoint since $\{6, 7\} \cap \{7, 8\} \neq \varnothing$

Lecture 24 - 3/24/2014

Cartesian product of sets

The ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.
For example $(3, 8) \neq (8, 3)$ and $(1, 1) = (1, 1)$.

Definition 15 (Cartesian Product)

Let A and Bbe sets. The **cartesian product** of A and B is the set
$$A \times B = \{(a, b) | \, a \in A \text{ and } b \in B\}$$

Example 16

Let $A = \{1, 3\}$ and $B = \{3, 5, 6\}$. Then
$$A \times B = \{(1, 3), (1, 5), (1, 6), (3, 3), (3, 5), (3, 6)\}$$

Nathan Broaddus Fundamentals of Higher Math

Cartesian product of sets

The ordered n-tuples $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ are equal if and only if $a_1 = b_1$ and $a_2 = b_2$ and $\cdots$ and $a_n = b_n$.
For example $(3, 12, 5, 5) \neq (3, 5, 12, 5)$ and $(1, 1, 8) = (1, 1, 8)$.

Definition 17 (Cartesian Product)

Let A_1, A_2, $\cdots$, A_n be sets. The **cartesian product** of A_1, A_2, $\cdots$ and A_n is the set

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) | (\forall i \in \mathbf{N})(1 \leq i \leq n \Rightarrow a_i \in A_i)\}$$

Example 18

Let $A = \{1, 3\}$. Then

$$A \times A \times A = \{(1,1,1), (1,1,3), (1,3,1), (1,3,3), (3,1,1), (3,1,3), (3,3,1), (3,3,3)\}$$