Math 3345
Fundamentals of Higher Mathematics

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Course Info

HW23 Due Friday, April 4, 2014

- Section 11 Exercises 4f, 7, 11

Warm-up Problems

Problem 1

Solution to Section 10 Exercise 33a:

Show that if A, B, C and D are sets then

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

The Empty Function

Question 1

Are there any functions f such that $f : \mathbf{N} \to \varnothing$?

Answer

If $f : \mathbf{N} \to \varnothing$ then by the definition of function $f \subset \mathbf{N} \times \varnothing = \varnothing$. Hence f must be a subset of $\varnothing$. Then only subset of $\varnothing$ is $\varnothing$ so we must have $f = \varnothing$. Now we must check that $f = \varnothing$ satisfies the following condition:

$$(\forall a \in \mathbf{N})(\exists b \in \varnothing)\,(a, b) \in f.$$

$(\exists b \in \varnothing)\,(a, b) \in f$ is always false so $(\forall a \in \mathbf{N})(\exists b \in \varnothing)\,(a, b) \in f$ is false. Thus $f = \varnothing$ is not a function from $\mathbf{N}$ to $\varnothing$. Thus **there are no such functions**.

The Empty Function

Question 2

Are there any functions f such that $f : \varnothing \to \mathbf{N}$?

Answer

If $f : \varnothing \to \mathbf{N}$ then by the definition of function $f \subset \varnothing \to \mathbf{N} = \varnothing$. Hence again f must be a subset of $\varnothing$ so we must have $f = \varnothing$. Now we must check that $f = \varnothing$ satisfies the following condition:

$$(\forall a \in \varnothing)(\exists b \in \mathbf{N})\,(a, b) \in f.$$

$(\forall a \in \varnothing)(\exists b \in \mathbf{N})\,(a, b) \in f$ is vacuously true. Thus $f = \varnothing$ is a function from $\varnothing$ to N. Thus $f = \varnothing$ **is the unique function from** $\varnothing$ **to** $\mathbf{N}$.

The Empty Function

In Question 2 above the set $\mathbf{N}$ played no role so the same reasoning shows that for any set A there is a unique function from $\varnothing$ to A.

Definition 2 (Empty Function)

If A is a set then $f = \varnothing$ is the unique function $f : \varnothing \to A$. f is called **empty function** and is denoted

$$\varnothing : \varnothing \to A.$$

In Question 1 above only used fact that $\mathbf{N}$ is nonempty so same reasoning proves:

Theorem 3

If A is a nonempty set then there are no functions f such that $f : A \to \varnothing$. If $A = \varnothing$ then we have the empty function $\varnothing : \varnothing \to \varnothing$.

Functional Equations

Suppose $h : \mathbf{R} \to \mathbf{R}$ and h satisfies

$$\forall x, y \in \mathbf{R} \quad h(x+y) = h(x) + h(y) \tag{1}$$

Then

Claim 5

$h(0) = 0$.

Proof.

$$h(0) = h(0+0) = h(0) + h(0).$$

Hence

$$h(0) = h(0) + h(0).$$

Thus

$$0 = h(0).$$

□

Functional Equations

Suppose $h : \mathbf{R} \to \mathbf{R}$ and h satisfies

$$\forall x, y \in \mathbf{R} \quad h(x+y) = h(x) + h(y)$$

Then

Claim 6

$h(-x) = -h(x)$.

Proof.

Let $x \in \mathbf{R}$. Then

$$0 = h(0) = h(x + (-x)) = h(x) + h(-x).$$

Hence

$$0 = h(x) + h(-x).$$

Thus

$$-h(x) = h(-x).$$

□

Functional Equations

Suppose $h : \mathbf{R} \to \mathbf{R}$ and h satisfies

$$\forall x, y \in \mathbf{R} \quad h(x+y) = h(x) + h(y)$$

Then

Claim 7

For all $n \in \omega$ we have $h(nx) = n \cdot h(x)$.

Proof.

Let $x \in \mathbf{R}$. Let $P(n)$ be the statement "$h(nx) = n \cdot h(x)$."
Base Case $h(0x) = h(0) = 0 = 0 \cdot h(x)$. So $P(0)$ holds. Inductive step
Suppose $n \in \omega$ and $P(n)$ is true.
Then

$$h((n+1)x) = h(nx + x) = h(nx) + h(x) = nh(x) + h(x) = (n+1)h(x).$$

Thus by induction $(\forall n \in \omega)\, h(nx) = n \cdot h(x)$. □

Composition of functions

Definition 4 (Composition of functions)

Let f and g be functions. The **composition** of the function g with the function f is the function $g \circ f$ with domain

$$\text{Dom}(g \circ f) = \{x \in \text{Dom}(f) | \, f(x) \in \text{Dom}(g)\}$$

given by the formula $g \circ f(x) = g(f(x))$.

Example 5 (Composition of functions)

Let $f(x) = x^2$ and $g(x) = x + 2$. Then $g \circ f(x) = x^2 + 2$ and $f \circ g(x) = (x+2)^2$ where $\text{Dom}(g \circ f) = \text{Dom}(f \circ g) = \mathbf{R}$.

Example 6 (More compositions of functions)

1.

$$f(x) = \sqrt{x}$$
$$g(x) = \sin(x)$$

Then

$$g \circ f(x) = \sin\sqrt{x} \qquad \mathrm{Dom}(g \circ f) = [0, \infty)$$
$$f \circ g(x) = \sqrt{\sin(x)} \qquad \mathrm{Dom}(f \circ g) = \bigcup_{n \in \mathbf{Z}} [2n\pi, (2n+1)\pi]$$

2. Let $f(x) = -1$ and $g(x) = \sqrt{x}$. Then

$$\begin{aligned} \mathrm{Dom}\, g \circ f &= \{x \in \mathbf{R} \mid f(x) \in \mathrm{Dom}(g)\} \\ &= \{x \in \mathbf{R} \mid -1 \in [0, \infty)\} \\ &= \varnothing \end{aligned}$$

so $g \circ f$ is the **empty function** $\varnothing : \varnothing \to \mathbf{R}$