Math 3345
Fundamentals of Higher Mathematics

Nathan Broaddus

Ohio State University

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Course Info

HW23 Due Friday, April 4, 2014

▶ Section 11 Exercises 4f, 7, 11
Warm-up Problems

Problem 1

Solution to Section 10 Exercise 33a:

Show that if $A$, $B$, $C$ and $D$ are sets then

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

The Empty Function

Question 1

Are there any functions $f$ such that $f : N \rightarrow \emptyset$?

Answer

If $f : N \rightarrow \emptyset$ then by the definition of function $f \subset N \times \emptyset = \emptyset$. Hence $f$ must be a subset of $\emptyset$. Then only subset of $\emptyset$ is $\emptyset$ so we must have $f = \emptyset$. Now we must check that $f = \emptyset$ satisfies the following condition:

$$(\forall a \in N) (\exists b \in \emptyset) (a, b) \in f.$$ 

$(\exists b \in \emptyset) (a, b) \in f$ is always false so $(\forall a \in N) (\exists b \in \emptyset) (a, b) \in f$ is false. Thus $f = \emptyset$ is not a function from $N$ to $\emptyset$. Thus there are no such functions.
The Empty Function

Question 2

Are there any functions $f$ such that $f : \emptyset \rightarrow \mathbb{N}$?

Answer

If $f : \emptyset \rightarrow \mathbb{N}$ then by the definition of function $f \subset \emptyset \rightarrow \mathbb{N} = \emptyset$. Hence again $f$ must be a subset of $\emptyset$ so we must have $f = \emptyset$. Now we must check that $f = \emptyset$ satisfies the following condition:

$$(\forall a \in \emptyset)(\exists b \in \mathbb{N})(a, b) \in f.$$  

$(\forall a \in \emptyset)(\exists b \in \mathbb{N})(a, b) \in f$ is vacuously true. Thus $f = \emptyset$ is a function from $\emptyset$ to $\mathbb{N}$. Thus $f = \emptyset$ is the unique function from $\emptyset$ to $\mathbb{N}$.

The Empty Function

In Question 2 above the set $\mathbb{N}$ played no role so the same reasoning shows that for any set $A$ there is a unique function from $\emptyset$ to $A$.

Definition 2 (Empty Function)

If $A$ is a set then $f = \emptyset$ is the unique function $f : \emptyset \rightarrow A$. $f$ is called empty function and is denoted

$$\emptyset : \emptyset \rightarrow A.$$  

In Question 1 above only used fact that $\mathbb{N}$ is nonempty so same reasoning proves:

Theorem 3

If $A$ is a nonempty set then there are no functions $f$ such that $f : A \rightarrow \emptyset$. If $A = \emptyset$ then we have the empty function $\emptyset : \emptyset \rightarrow \emptyset$. 


Functional Equations

Suppose $h : \mathbb{R} \to \mathbb{R}$ and $h$ satisfies

$$\forall x, y \in \mathbb{R} \quad h(x + y) = h(x) + h(y) \quad (1)$$

Then

Claim 5

$h(0) = 0$.

Proof.

$$h(0) = h(0 + 0) = h(0) + h(0).$$

Hence

$$h(0) = h(0) + h(0).$$

Thus

$$0 = h(0).$$

Claim 6

$h(-x) = -h(x)$.

Proof.

Let $x \in \mathbb{R}$. Then

$$0 = h(0) = h(x + (-x)) = h(x) + h(-x).$$

Hence

$$0 = h(x) + h(-x).$$

Thus

$$-h(x) = h(-x).$$
Functional Equations

Suppose $h : \mathbb{R} \to \mathbb{R}$ and $h$ satisfies

\[ \forall x, y \in \mathbb{R} \quad h(x + y) = h(x) + h(y) \]

Then

**Claim 7**

For all $n \in \omega$ we have $h(nx) = n \cdot h(x)$.

**Proof.**

Let $x \in \mathbb{R}$. Let $P(n)$ be the statement "$h(nx) = n \cdot h(x)$.”

- **Base Case** $h(0x) = h(0) = 0 = 0 \cdot h(x)$. So $P(0)$ holds.
- **Inductive step**

Suppose $n \in \omega$ and $P(n)$ is true. Then

\[ h((n + 1)x) = h(nx + x) = h(nx) + h(x) = nh(x) + h(x) = (n + 1)h(x). \]

Thus by induction ($\forall n \in \omega$) $h(nx) = n \cdot h(x)$. 

Composition of functions

**Definition 4 (Composition of functions)**

Let $f$ and $g$ be functions. The composition of the function $g$ with the function $f$ is the function $g \circ f$ with domain

\[ \text{Dom}(g \circ f) = \{ x \in \text{Dom}(f) | f(x) \in \text{Dom}(g) \} \]

given by the formula $g \circ f(x) = g(f(x))$.

**Example 5 (Composition of functions)**

Let $f(x) = x^2$ and $g(x) = x + 2$. Then $g \circ f(x) = x^2 + 2$ and $f \circ g(x) = (x + 2)^2$ where $\text{Dom}(g \circ f) = \text{Dom}(f \circ g) = \mathbb{R}$. 
Example 6 (More compositions of functions)

1. 

\[ f(x) = \sqrt{x} \]
\[ g(x) = \sin(x) \]

Then

\[ g \circ f(x) = \sin \sqrt{x} \quad \text{Dom}(g \circ f) = [0, \infty) \]
\[ f \circ g(x) = \sqrt{\sin(x)} \quad \text{Dom}(f \circ g) = \bigcup_{n \in \mathbb{Z}} [2n\pi, (2n+1)\pi] \]

2. Let \( f(x) = -1 \) and \( g(x) = \sqrt{x} \). Then

\[ \text{Dom} g \circ f = \{x \in \mathbb{R} \mid f(x) \in \text{Dom}(g)\} \]
\[ = \{x \in \mathbb{R} \mid -1 \in [0, \infty)\} \]
\[ = \emptyset \]

so \( g \circ f \) is the empty function \( \emptyset : \emptyset \to \mathbb{R} \)